Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH712P / MTH734U: Topics in Probability and Stochastic Processes

Duration: 3 hours

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [25 marks]. Let $(N_1(t),\ t \geq 0)$ and $(N_2(t),\ t \geq 0)$ be two independent Poisson processes with intensities λ_1 and λ_2, respectively.

(a) Calculate the covariance $\operatorname{Cov}\big(N_1(t) + N_2(t), N_1(t) - N_2(t)\big)$. **[5]**

(b) Given that $N_1(2) = 9$ and $N_2(2) = 6$, find the probability that the N_1-process reaches level 10 before the N_2-process does. **[5]**

(c) For given n_1 and n_2, what is the probability that there exists time t such that $N_1(t) = n_1$ and $N_2(t) = n_2$? **[5]**

(d) Consider the Markov chain $D(t) = N_1(t) - N_2(t),\ t \geq 0$.

(i) Find the transition rates of this Markov chain. **[5]**

(ii) For given positive integers a and b determine the probability that the Markov chain reaches state b before state $-a$. **[5]**

Question 2 [25 marks]. Attacks on a data network occur according to a Poisson process with rate λ. Every attack is eventually detected. The time between occurrence of an attack and its detection has exponential distribution with parameter μ, and these times for different attacks are independent.
Consider the Markov chain $(X(t),\ t \geq 0)$, where $X(t)$ is the number of undetected attacks at time t, and suppose that $X(0) = 0$.

(a) Determine $\mathbb{E}X(t)$. **[6]**

(b) Determine the transition rates of this Markov chain, and write down the first three rows of the generator matrix. **[6]**

(c) Show that the distribution

$$\pi_k = e^{-\rho}\,\frac{\rho^k}{k!}, \quad k = 0, 1, \ldots,$$

with $\rho := \lambda/\mu$ is a stationary distribution of this Markov chain. **[7]**

(d) For t large, what is the approximate probability that there are three undetected attacks at time t? Explain your answer. **[6]**

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Question 3 [25 marks]. The lifespan of a smartphone until it breaks down is a random variable which has exponential distribution with parameter λ. A smartphone is replaced with a new one as soon as it breaks down or its service age reaches value s, whichever happens earlier. Assume that a smartphone is new at time 0, and that the lifespans (until breaking down) of different smartphones are independent. Let W be the time of the first replacement which occurs because of the service age of a smartphone.

(a) What is the distribution of the number of smartphones that get replaced by time W? (Do not count 0 but count W as a replacement time.) Justify your answer. [**6**]

(b) Determine the cumulative distribution function of the time between two successive replacements. [**6**]

(c) Find $\mathbb{E}W$. [**7**]

(d) Find the long run replacement rate. [**6**]

Question 4 [25 marks]. In this question $(B(t),\ t \geq 0)$ is a standard Brownian motion.

(a) Compute $\mathbb{E}\left[B^2(3) - B^2(1)\right]$. [**5**]

(b) Determine $\mathbb{P}(B(s) > B(t))$ for $s > 0,\ t > 0,\ s \neq t$. [**5**]

(c) Show that $\mathbb{P}(B(1) > 0, B(2) > 0) > 1/4$. [**5**]

(d) Is the process $W(t) = -3\,B(9t)$ a standard Brownian motion? Explain your answer. [**5**]

(e) For the process $X(t) = e^{-t}B(e^{2t})$ calculate the covariance function

$$c(s,t) = \mathrm{Cov}(X(s), X(t))$$

for $s \geq 0,\ t \geq 0$.

[**5**]

End of Paper.

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