Queen Mary
University of London

Main Examination period 2017

MTH734U / MTHM012 / MTH712P: Topics in Probability and Stochastic Processes

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: Dr Dudley Stark, Dr Christopher Joyner

Question 1. [20 marks] Let $N(t)$, $t \geq 0$, be a continuous time renewal process with interoccurrence times $X_i > 0$ for $i = 1, 2, \ldots$, which are independent, identically distributed continuous random variables with common distribution $\mathbb{P}(X_i \leq x) = F(x)$. Let $S_0 = 0$ and let $S_n = X_1 + X_2 + \cdots + X_n$ be the waiting time until the occurrence of the nth event for $n \geq 1$. Suppose $\mu = \mathbb{E}(X_1) < \infty$. Let $M(t) = \mathbb{E}(N(t))$.

(a) Prove that for all integers $n \geq 1$ and all real numbers $t > 0$,

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t).$$

[**5**]

(b) Let X_i take on positive integer values. Show that, with $p_i = \mathbb{P}(X_1 = i)$, the renewal function $M(n)$ satisfies

$$M(n) = F_(n) + \sum_{i=1}^{n-1} p_i M(n-i).$$

[**6**]

(c) Suppose that each X_i is Geometric(β) distributed with probability mass function $\mathbb{P}(X_i = k) = \beta(1-\beta)^{k-1}$, $k = 1, 2, \ldots$, for a parameter $\beta \in [0, 1]$.

(i) Use the recursive formula in (b) to find $M(1)$, $M(2)$ and $M(3)$. [**3**]

(ii) State the distribution of $N(n)$, justifying your answer by a short argument. [**3**]

(iii) Either by using the disribution of $N(n)$ or by using induction, prove a formula for all $M(n)$, $n \geq 1$. [**3**]

Question 2. [20 marks]

(a) Given a semi-Markov process on states $\{1, 2, \ldots, N\}$, suppose that when the process enters state i, it stays there a random amount of time having expectation μ_i after which it jumps to state j with probability $P_{i,j}$.

(i) Let π_i denote the proportion of transitions to i in the long run. Write down the equations derived in a lecture which, when they can be solved uniquely, determine the π_i. [**3**]

(ii) State an example of a semi-Markov process for which unique π_i do not exist. Find all solutions for the equations for the π_i for your example. [**6**]

(b) (i) A particular machine in a factory is powered by a battery. The battery is in constant use. As soon as the battery in use fails, it is replaced with a new battery. If the lifetime of a battery (in hours) is distributed uniformly over the interval $(30, 60)$, then at what rate in the long run are batteries replaced? [**3**]

(ii) Suppose that the lifetime of a battery (in hours) is still distributed uniformly over the interval $(30, 60)$, but that now each time a failure occurs a worker must go and get a new battery from storage, after which the failed battery is immediately replaced with the new battery. If the amount of time (in hours) it takes a worker to get a new battery is uniformly distributed over $(0, 1)$, then what is the new rate at which batteries are replaced in the long run? For what proportion time is the battery in the machine a failed battery? [**8**]

Question 3. [20 marks] Let S_i for $i = 1, 2, \ldots$ denote the time of the ith event of a Poisson process $N(t)$, $t \geq 0$, with rate $\theta > 0$.

(a) Find $\mathbb{E}(S_i)$. **[4]**

(b) Find the Laplace transform $\hat{F}_{S_i}(\lambda)$ of S_i. **[6]**

(c) Derive

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} S_i \,\middle|\, N(t) = n\right).$$

[10]

Question 4. [20 marks]

(a) Let $X(t)$ be a continuous time Markov chain with conditional probability densities

$$f_n(y_n, t_n | y_{n-1}, t_{n-1}; y_{n-2}, t_{n-2}; \ldots; y_1, t_1),$$

where $0 \leq t_1 < t_2 < \cdots < t_n$ and $y_i \in \mathbb{R}$ for all $1 \leq i \leq n$, where $\mathbb{R}$ is the set of real numbers. State what is meant by the Markov property for $X(t)$. **[3]**

(b) In the East London Health Club there are two swimmers who are training for the Olympics. Each swimmer alternates between a period of swimming freestyle, a period of swimming the backstroke, another period of swimming freestyle, and so on, for a long period of time. The lengths of the periods of swimming freestyle are all exponentially distributed with mean of 5 minutes and the lengths of the periods of swimming backstroke are all exponentially distributed with mean of 4 minutes. The lengths of the periods are all independent of each other.

(i) Let $X(t)$ be the number of swimmers swimming the backstroke at time $t > 0$. What is the generator $\mathbf{G}$ for the continuous time Markov chain $X(t)$? **[9]**

(ii) Use the generator $\mathbf{G}$ to find the limiting distribution of $X(t)$. In the long run, for what proportion of time is one swimmer swimming freestyle and the other swimmer swimming the backstroke? **[8]**

Question 5. [20 marks] Let $B(t)$ be a standard Brownian motion with $B(0) = 0$.

(a) State what is meant by the independent increments property. [**3**]

(b) Determine the distribution of $B(s) + B(t)$. [**5**]

(c) Let $\alpha_1, \ldots, \alpha_n$ be real constants. Prove that

$$\sum_{i=1}^{n} \alpha_i B(t_i)$$

is normally distributed with mean zero and variance

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \min(t_i, t_j).$$

[**5**]

(d) Reflecting Brownian motion $R(t)$ is defined by $R(t) = |B(t)|$.

(i) Let $y \geq 0$ be given. Show that

$$\mathbb{P}(R(t) < y) = \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{-y}{\sqrt{t}}\right),$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$ is the cumulative distribution function of the standard normal distribution. [**4**]

(ii) A Gaussian process $X(t)$ is one for which the distribution of $X(t)$ is normally distributed for all $t > 0$. For example, $B(t)$ is a Gaussian process. Is $R(t)$ a Gaussian process? Give a brief explanation of your answer. [**3**]

End of Paper.