Queen Mary University of London

Main Examination period 2018

MTH714U / MTHM024: Group Theory

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: L. H. Soicher and M. Fayers

Question 1. Let G be a group and let Ω be a set.

(a) What is meant by an **action** of G on Ω? **[3]**

(b) Suppose that G acts on Ω, and let $\alpha \in \Omega$.

(i) What is meant by the **orbit** $\mathrm{Orb}_G(\alpha)$ of α, and what is meant by the **stabiliser** $\mathrm{Stab}_G(\alpha)$ of α? What does it mean to say that G acts **transitively** on Ω? **[6]**

(ii) Prove that $\mathrm{Stab}_G(\alpha)$ is a subgroup of G. **[5]**

(iii) Let $g \in G$ and let $H = \mathrm{Stab}_G(\alpha)$. Prove that $\mathrm{Stab}_G(\alpha g) = g^{-1}Hg$. **[6]**

(iv) Deduce that if G acts transitively on Ω then the set $\{\mathrm{Stab}_G(\beta) : \beta \in \Omega\}$ is a conjugacy class of subgroups of G. **[5]**

Question 2. Let G be a group.

(a) Define what is meant by the **commutator** $[g, h]$ of elements $g, h \in G$, and what is meant by the **commutator subgroup** (or **derived group**) G' of G. What is meant by a **normal subgroup** of G and what is meant by a **maximal subgroup** of G? **[8]**

(b) Prove that if N is a normal subgroup of G such that G/N is abelian, then G' is a subgroup of N. **[5]**

(c) Suppose G acts primitively on a set Ω, and let N be a normal subgroup of G. Prove that either N acts trivially on Ω (that is, N lies in the kernel of the action), or N acts transitively on Ω. **[6]**

(d) Suppose G has a faithful primitive action on a set Ω, let $\alpha \in \Omega$, and suppose that there is an abelian normal subgroup A of $\mathrm{Stab}_G(\alpha)$ with the property that the G-conjugates of A generate G.

Prove that every non-trivial normal subgroup of G contains the commutator subgroup G'. [In other words, you are to prove Iwasawa's Lemma. You may assume, without proof, that since G acts primitively on Ω, $\mathrm{Stab}_G(\alpha)$ is a maximal subgroup of G.] **[7]**

Question 3.

(a) What is meant by a **permutation** of $\{1, \ldots, n\}$, what is meant by an **even** permutation of $\{1, \ldots, n\}$, what is meant by the **alternating group** A_n, and what does it mean to say that a group G is **simple**? **[8]**

(b) Prove that the alternating group A_5 is simple. [You may state, without proof, the sizes of the conjugacy classes of the group A_5.] **[7]**

(c) Prove that the alternating group A_n is simple, for all $n \geq 5$. [You may assume, without proof, that the group A_n acts primitively on $\{1, \ldots, n\}$.] **[9]**

Question 4.

(a) Let F be a field, let $n > 1$, let $V = F^n$, and let a be a non-zero vector in V.

(i) Define what is meant by a **transvection** $T(a, f)$ on V, and what is meant by the **transvection group** $A(a)$. **[4]**

(ii) Prove that if $g \in \mathrm{GL}(n, F)$ then $g^{-1}A(a)g = A(ag)$. **[6]**

(b) For each group

$$\mathrm{GL}(2, 7),\ \mathrm{SL}(2, 7),\ \mathrm{PGL}(2, 7),\ \mathrm{PSL}(2, 7),$$

determine the order of the group and whether the group is simple. [You should briefly justify your answers.] **[10]**

(c) Determine a composition series for $\mathrm{GL}(2, 4)$, and the corresponding list of composition factors. [You are not required to justify your answers.] **[5]**

End of Paper.