Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH709U / MTH776P: Bayesian Statistics

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Only approved non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: A. Shestopaloff, M. Iacopini

Question 1 [12 marks].

(a) Show that the exponential distribution with mean θ^{-1} is an exponential family distribution **[6]**

(b) Identify the natural parameter. **[3]**

(c) If a random sample of n exponentially distributed observations is collected write down a one-dimensional sufficient statistic for θ based on n observations. **[3]**

Question 2 [25 marks]. Let $x_1, \ldots, x_n$ be a random sample from a Uniform$(0, \theta)$ distribution.

(a) Suppose that we put a Pareto prior on θ with density function

$$p(\theta) = \frac{\alpha \theta_0^\alpha}{\theta^{\alpha+1}}$$

for $\theta \geq \theta_0$ and $\alpha > 0$. Find the posterior density of θ, including all appropriate normalizing constants. **[10]**

(b) Let θ_{true} be the true value of θ. Describe what happens to the posterior density of θ as $n \to \infty$. **[5]**

(c) Find the optimal estimator of θ under the loss function

$$L(t, \theta) = \frac{(t-\theta)^2}{t}$$

where t is an estimator of θ. **[10]**

Question 3 [23 marks].

Consider a random sample $x_1, \ldots, x_n$ from a distribution with density

$$p(x|\theta) = \theta \exp(-x\theta)$$

with $x \geq 0$ and $\theta > 0$.

(a) What is the *Jeffreys' prior* for θ? **[10]**

(b) Suppose we reparametrize the model via $\mu = 1/\theta$. Derive the Jeffreys' prior in this new parametrization. **[8]**

(c) Transform the Jeffreys' prior you have obtained in a) to the μ parametrization used in b). How does it compare to Jeffreys' prior you have computed in b)? **[5]**

Question 4 [15 marks].

(a) Consider a random sample $x_1, \dots x_n$ from a Bernoulli(θ) distribution. We observe

$$x_1 = x_2 = \cdots = x_n = 0.$$

We would like to compare two Bayesian models for this data.

$$H_0 : \theta = 1/2 \quad \text{versus} \quad H_1 : \theta \sim \text{Uniform}(0, 1).$$

We assume equal prior probabilities for H_0 and H_1 ie., $\pi_0 = \pi_1 = 1/2$. Find the posterior probability α_0 of H_0 as a function of n. **[10]**

(b) Define a $100(1-\alpha)\%$ credible interval and a $100(1-\alpha)\%$ Highest Posterior Density (HPD) interval for a parameter θ. What extra property does an HPD interval have? **[5]**

Question 5 [25 marks]. The two-stage linear model is given by

$$\begin{aligned} y|\theta_1, C_1, A_1 &\sim N(A_1\theta_1, C_1) \\ \theta_1|\mu, C_2 &\sim N(\mu, C_2) \end{aligned}$$

where A_1, C_1, C_2 and μ are known.
A researcher is interested in analyzing the commute time of workers at a company in Toronto as a function of the distance they live from the workplace. The assumption is that commute time y (in minutes) is linearly related to the distance x (in km). Five workers are surveyed and the commute time and distance are recorded.

x	y
25	40
11	19
40	65
33	55
6	10

The simple linear regression model

$$y_i = \alpha + \beta(x_i - \bar{x}) + \varepsilon_i \quad i = 1, \dots, 5$$

is appropriate for these data. Assume that the prior for α is normal with mean 0 and variance 1, the prior for β is normal with mean 1.5 and variance 0.2, α and β are independent and the ε_i are independent, normally distributed with mean zero and variance 1.

(a) Show that this model can be written as a two-stage linear model and hence find the posterior distributions of α and β. **[15]**

(b) Are the posterior distributions of α and β independent? Justify your answer. **[5]**

(c) Describe how you would use Gibbs sampling to draw from the posterior distribution of (α, β). **[5]**

End of Paper – An appendix of 3 pages follows.

Bayesian Statistics – Common Distributions

Discrete Distributions

Distribution	Density	Range of Variates	Mean	Variance
Uniform	$\frac{1}{N}$	$N = 1, 2, \ldots$ $x = 1, 2, \ldots, N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$
Bernoulli	$p^x(1-p)^{1-x}$	$0 \le p \le 1,\ x = 0, 1$	p	$p(1-p)$
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	$0 \le p \le 1,\ n = 1, 2, \ldots$ $x = 0, 1, \ldots n$	np	$np(1-p)$
Poisson	$\frac{\exp(-\lambda)\lambda^x}{x!}$	$\lambda > 0,\ x = 0, 1, 2, \ldots$	λ	λ
Geometric	$p(1-p)^x$	$0 < p \le 1,\ x = 0, 1, 2, \ldots$	$\frac{(1-p)}{p}$	$\frac{(1-p)}{p^2}$
Negative Binomial	$\binom{r+x-1}{x} p^r (1-p)^x$	$0 < p \le 1,\ r > 0$ $x = 0, 1, 2, \ldots$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$

Continuous Distributions

Distribution	Density	Range of Variates	Mean	Variance
Uniform	$\frac{1}{b-a}$	$-\infty < a < b < \infty$ $a < x < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal $N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}} \exp[\frac{-(x-\mu)^2}{2\sigma^2}]$	$-\infty < \mu < \infty$ $\sigma > 0,\ -\infty < x < \infty$	μ	σ^2
Normal $No(\mu, h)$	$\frac{\sqrt{h}}{\sqrt{2\pi}} \exp[\frac{-h(x-\mu)^2}{2}]$	$-\infty < \mu < \infty$ $h > 0,\ -\infty < x < \infty$	μ	h^{-1}
Exponential	$\lambda \exp(-\lambda x)$	$\lambda > 0,\ x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma $Ga(\alpha, \beta)$	$\frac{\beta^\alpha x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)}$	$\alpha > 0,\ \beta > 0,\ x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Inverse-gamma $IGa(\alpha, \beta)$	$\frac{\beta^\alpha x^{-\alpha-1} \exp(-\beta/x)}{\Gamma(\alpha)}$	$\alpha > 2,\ \beta > 0,\ x > 0$	$\frac{\beta}{\alpha-1}$	$\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$

Distribution	Density	Range of Variates	Mean	Variance
Beta $Be(a,b)$	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$	$a>0,\ b>0,\ 0<x<1$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$
$t_\nu(m,g)$	$\frac{g^{1/2}\Gamma((\nu+1)/2)}{\sqrt{(\nu\pi)}\Gamma(\nu/2)}$	$-\infty<x<\infty$	location m	precision g,
	$\times\left[1+\frac{g}{\nu}(x-m)^2\right]^{-(\nu+1)/2}$	dof ν		
F_n^m	$\frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{\frac{m}{2}}$	$m,n=1,2,\ldots$	$\frac{n}{n-2}$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$
	$\times\frac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}}$	$x\geq 0$	for $n>2$	for $n>4$
χ_k^2	$\frac{1}{\Gamma(k/2)2^{k/2}}x^{k/2-1}\exp(-\frac{x}{2})$	$k=1,2,\ldots,\ x>0$	k	$2k$
Pareto	$\frac{\alpha\beta^\alpha}{x^{\alpha+1}}$	$\alpha>0,\beta>0,x>\beta$	$\frac{\beta\alpha}{(\alpha-1)}$	$\frac{\beta^2\alpha}{(\alpha-1)^2(\alpha-2)}$

End of Appendix.