

Main Examination period 2023 – May/June – Semester B

## MTH6142: Complex Networks

## Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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## Question 1 [50 marks].

Consider the undirected network G = (V, E) with N = 15 nodes described by the following set of links:

$$E = \{(1,2), (1,3), (2,4), (2,5), (3,6), (3,7), (4,8), (4,9), (5,10), (5,11), (6,12), (6,13), (7,14), (7,15)\}.$$

- (a) Calculate the number of links L in the network, and draw the network. Is the network connected? Which type of network is this? (*Explain your answer*). [7]
- (b) Write down the degree sequence, the degree distribution and the average degree of the network G. [7]
- (c) Compute the diameter D of the network G, and list all the pairs of nodes which are at distance D (*Hint: it is not necessary to compute all the node distances to* work out the diameter of the graph).
- (d) Evaluate the betweenness centrality for all the nodes in the network G. Use the following definition of betweenness centrality of a node *i*:

$$b_i = \sum_{r=1}^N \sum_{s=1}^N \frac{n_{rs}^i}{n_{rs}}$$

where  $n_{rs}$  is the number of shortest paths between node r and node s, while  $n_{rs}^i$  is the number of shortest paths between node r and node s passing through node i. *Hint:* (Notice that, according to this definition, paths starting from node i or ending in node j are also counted). [11]

- (e) Which are the nodes with the largest possible values of betweenness centrality in the network, and which are those with the largest values of closeness centrality and of degree centrality?
- (f) Suppose you can add a link to connect two nodes of the network G to obtain another network G'. List one link choice that will create a triangle in the network G', one link choice that will create a cycle of length 4, and link choice that will create a cycle of length 5. How many are all the possible link choices that will guarantee that the network G' has a cycle of length  $\ell = 7$ . [10]

**[6**]

**[9**]

**[6**]

8

[8]

## Question 2 [50 marks].

Consider the Erdös-Rènyi random graph ensemble  $\mathbb{G}_{N,L}$  and assume that the number of links grows as a function of N, as  $L(N) = \alpha N$ , where  $\alpha$  is a positive real constant (independent on N).

- (a) Write down the general expression, as a function of N and  $\alpha$ , for the probability of finding a given graph G = (V, E) in the ensemble  $\mathbb{G}_{N,L(N)}$ . [8]
- (b) Fix now N = 5 and consider the three cases  $\alpha_1 = 0.8$ ,  $\alpha_2 = 1$ , and  $\alpha_3 = 1.2$ . Evaluate, in each of the three cases, the probability of finding a given graph G = (V, E) with 5 nodes in the ensemble  $\mathbb{G}_{N,L(N)}$ .
- (c) Find a general expression for the average node degree  $\langle k \rangle$  in a graph of the ensemble  $\mathbb{G}_{N,L(N)}$ , and evaluate the average node degree  $\langle k \rangle_1$ ,  $\langle k \rangle_2$  and  $\langle k \rangle_3$  for the graphs in each of the three ensembles considered in point (b) (i.e. by fixing N = 5 and  $\alpha_1 = 0.8$ ,  $\alpha_2 = 1$ ,  $\alpha_3 = 1.2$  respectively). [7]
- (d) Find the function p(N) of the Erdös-Rènyi random graph ensemble  $\mathbb{G}_{N,p}$  which best approximate the Erdös-Rènyi random graph ensemble  $\mathbb{G}_{N,L(N)}$ . [5]
- (e) For the Erdös-Rènyi random graph ensemble  $\mathbb{G}_{N,p(N)}$  found in point (d), write down the distribution P(L) for the total number of links L in a graph. Calculate the variance of such distribution and find an expression for the ratio r between the standard deviation and the expectation value of L.
- (f) Consider the Erdös-Rènyi random graph ensemble  $\mathbb{G}_{N,p(N)}$  found in point (d), assume  $N \to \infty$ , and find the condition on  $\alpha$  for the graphs to have a giant component. Find the same result using the Molloy-Reed criterion.
- (g) Consider the Erdös-Rènyi random graph ensemble  $\mathbb{G}_{N,p(N)}$  found in point (d), assume  $N \to \infty$ , and find the expected numbers of triangles and 4-cliques as a function of  $\alpha$ . [8]

End of Paper.