Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6142: Complex networks

Duration: 2 hours

Date and time: 28th April 2015, 10:00-12:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): G. Bianconi

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Question 1 (40 marks).

Structural properties of a given network
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 7$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

- a) Is the network directed or undirected? (Give reasons) **[6]**
- b) Draw the network. **[6]**
- c) Write the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}, k_5^{in}, k_6^{in}, k_7^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}, k_5^{out}, k_6^{out}, k_7^{out}\}$. **[4]**
- d) Write the in-degree distribution of the network $P^{in}(k)$ and the out degree distribution of the network $P^{out}(k)$ for $k = 0, 1, 2, 3, 4, 5, 6$. **[4]**
- e) How many weakly connected components has the network? **[2]**
- f) List the nodes in each of the weakly connected components. **[2]**
- g) Does the network contain a strongly connected component? If yes, which nodes are part of the strongly connected component? **[2]**
- h) Define the eigenvector centrality $\mathbf{x}$. Calculate the eigenvector centrality $\mathbf{x}$ of the nodes of the network with adjacency matrix $\mathbf{A}$. **[10]**
- i) Is the result obtained in point h) expected? (Give reasons). **[4]**

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Question 2 (25 marks).

Diameter of a Cayley tree
A Cayley tree is a symmetric regular tree constructed starting from a central node of degree k (see figure 1 for an example of a Cayley tree with $k = 3$).
In a Cayley tree network every node at distance d from the central node has degree k until we reach the nodes at distance $\mathcal{P}$ that have degree one and are called the leaves of the network. In this question take $k = 3$.

- a) Show that the number of nodes at distance d from the central node, is $3 \times 2^{d-1}$ for $d \in [1, \mathcal{P}]$. [**5**]

- b) Using the properties of the geometric sum, show that the total number of nodes in the network is given by $N = 1 + 3\left[2^{\mathcal{P}} - 1\right]$. [**5**]

- c) Show that the diameter D is given by $D = 2\mathcal{P}$. [**2**]

- d) Using the results obtained in (b) and (c), express the diameter D of the network in terms of the total number of nodes N. [**5**]

- e) Consider the expression found in point (d).
Find the leading term of D in terms of the total number of nodes N in the network, in the limit $N \gg 1$. [**5**]

- f) Does the network display the small-world distance property? [**3**]

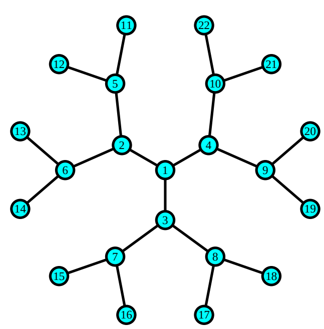

Figure 1: A Cayley tree network with $k = 3$ and $\mathcal{P} = 3$.

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Question 3 (35 marks).

Growing network model

The growing network models with preferential attachment generate networks that exhibit a power-law degree distribution. Consider the following model. At time $t = 0$ the network is formed by two nodes joined by a link.

- At every time step a single new node joins the network, so that at time t there will be exactly $N = 2 + t$ nodes. Every new node has initially $m = 1$ links.
- Each new link is attached to an existing node of the network. The target node i is chosen with probability Π_i following the modified preferential attachment rule $\Pi_i = \frac{k_i + A}{\sum_{j=1}^{N}(k_j + A)}$, where k_i is the degree of the node i and $A > -1$.

- a) What is the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i in the mean-field, continuous approximation? **[7]**

- b) What is the degree distribution of the network at large times in the mean-field approximation? **[7]**

- c) Using the degree distribution obtained in part (b) and assuming that the maximal degree of the network is $K = t^{1/(2+A)}$, calculate $\langle k^2 \rangle$ in the continuous approximation. **[7]**
  Note that $\langle \ldots \rangle$ indicates the average over the degree distribution of the network.

- d) For which values of A are the networks generated by this model scale-free?
  Hint: Comment on the limit of $\langle k^2 \rangle$ for $t \to \infty$. **[2]**

- e) Let $N(k)$ be average number of nodes with degree k.
  Write the master equation satisfied by $N(k)$. **[3]**

- f) Solve the master equation finding the exact results for the degree distribution $P(k)$ in the limit $N \to \infty$. **[9]**

End of Paper.

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