Queen Mary University of London

Main Examination period 2018

MTH6141/MTH6141P: Random Processes

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: David Ellis

Question 1. [13 marks] This question is about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw a transition graph for this Markov chain. **[3]**

(b) List the absorbing states of the Markov chain. **[2]**

(c) Using first-step analysis, or otherwise, find the probability that the Markov chain ever reaches state 5, given that $X_0 = 1$. Show your working. **[6]**

(d) Find the probability that the Markov chain ever reaches state 2, given that $X_0 = 1$. **[2]**

Question 2. [16 marks] This question is about a gambler. The gambler starts with an initial fortune of £2. He gambles once per day until he quits. He quits when either he loses all his money or he increases his fortune to £4. On a day when he has $\pounds j$, where $j \in \{1, 2, 3\}$, he gambles on the outcome of a game, and wins £1 with probability $(4 - j)/10$, but loses £1 with probability $(6 + j)/10$.

(a) The gambler's changing fortune can be modelled as a Markov chain with five states (of which two are absorbing states). Write down what these five states are. (In other words, write down which situations each of the five states corresponds to.) **[3]**

(b) Draw a transition graph for this Markov chain. **[4]**

(c) Write down a transition matrix for the Markov chain. (Indicate clearly which rows of the matrix correspond to which states.) **[3]**

(d) Using first-step analysis, or otherwise, find the expected number of gambles the gambler makes until he quits. Show your working. **[6]**

Question 3. [15 marks] Parts (a) and (b) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $S = \{1, 2, \ldots, n\}$ and transition matrix P.

(a) Let $w = (w_1, w_2, \ldots, w_n)$ be a probability vector. Define what it means for w to be an **equilibrium distribution** for the Markov chain, and define what it means for w to be a **limiting distribution** for the Markov chain. **[4]**

(b) Show that if w is a limiting distribution for the Markov chain, then w is an equilibrium distribution for the Markov chain. **[4]**

Parts (c) and (d) of this question are about a Markov chain $(Y_0, Y_1, Y_2, \ldots)$ with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

(c) Is this Markov chain regular? Justify your answer. **[4]**

(d) It can be checked that the unique equilibrium distribution of this Markov chain is $(7/57, 7/19, 4/19, 17/57)$. Does the Markov chain have a limiting distribution? If so, what is it? Justify your answer briefly. (You may quote any result from the course without proof.) **[3]**

Question 4. [15 marks] This question is about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(a) Is this Markov chain irreducible? Justify your answer carefully. **[3]**

(b) Find the equilibrium distribution of the Markov chain. **[6]**

(c) Prove that this Markov chain has no limiting distribution. **[4]**

(d) Approximately what proportion of time does this Markov chain spend in the state 1, in the long run? **[2]**

Question 5. [16 marks] Parts (a) and (b) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space S.

(a) Define what it means for a state $i \in S$ to be **transient**, in terms of the return probability $f_{i,i}$. **[2]**

(b) State a condition, in terms of the t-step transition probabilities $p_{i,i}^{(t)}$ alone, which is equivalent to the state i being transient. **[2]**

Parts (c)-(e) of this question are about a Markov chain with state space $\mathbb{Z}$ and transition probabilities

$$p_{i,i+1} = \tfrac{1}{8} \text{ for all } i \in \mathbb{Z}, \quad p_{i,i-1} = \tfrac{7}{8} \text{ for all } i \in \mathbb{Z}, \quad p_{i,j} = 0 \text{ for all other } i, j \in \mathbb{Z}.$$

(c) Show that

$$p_{0,0}^{(t)} = \begin{cases} \binom{t}{t/2}\left(\frac{7}{64}\right)^{t/2} & \text{if } t \text{ is even;} \\ 0 & \text{if } t \text{ is odd.} \end{cases}$$

[4]

(d) Using part (b) and part (c), or otherwise, show that the state 0 is transient. You may assume that

$$\binom{2n}{n} \leq 2^{2n} \quad \forall n \in \mathbb{N}.$$

[4]

(e) Is the state -100 recurrent or transient? Justify your answer briefly. **[4]**

Question 6. [**25 marks**]

(a) Let $(X(t) : t \geq 0)$ be a continuous-time stochastic process, and let $\lambda > 0$. Define what it means for $(X(t) : t \geq 0)$ to be a **Poisson process of rate** λ. [**5**]

Parts (b)-(f) of this question are about trains arriving at a station platform. Virgin trains arrive at the platform according to a Poisson process with rate 4 per hour. West Midlands trains arrive at the platform according to a Poisson process with rate 6 per hour. These two Poisson processes are independent of one another. Your answers to parts (b)-(f) should be expressed in terms of powers of e (where necessary), but they should be simplified in all other ways.

(b) Suppose I arrive on the platform at 9 am with a ticket valid on Virgin trains only. What is the probability that I have to wait more than half an hour for the arrival of a Virgin train? [**3**]

(c) Suppose each train that arrives is full with probability 1/4, independently of all other trains. Suppose again that I arrive on the platform at 9 am with a ticket valid on Virgin trains only. What is the probability that I have to wait more than half an hour for the arrival of a Virgin train that is not full? [**4**]

(d) Suppose instead that I arrive on the platform at 9 am with a ticket valid on both Virgin and West Midlands trains. What is the probability that I have to wait more than half an hour before the arrival of a train of either kind? [**3**]

(e) Suppose that exactly three Virgin trains arrived between 9 am and 10 am. Conditional on this information, what is the probability that exactly one Virgin train arrived between 9 am and 9.15 am? [**4**]

(f) Suppose again that exactly three Virgin trains arrived between 9 am and 10 am. Conditional on this information, what is the expected time of arrival of the first Virgin train (after 9 am)? Show your working. [**6**]

End of Paper.