Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH6140 / MTH6140P: Linear Algebra II

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: S. Majid, I. Morris

You may refer to general results from lectures.

Question 1 [20 marks]. In this question, $V = \mathbb{K}[x]_3$ denotes the vector space consisting of zero or polynomials of degree ≤ 3, with coefficients in a field $\mathbb{K}$. Let

$$U = \{f \in V \mid f(0) = 0,\ f(2) = 0\}.$$

(a) Show that U is a subspace of V. **[6]**

In the rest of this question, let $u_1 = x(x-2)$, $\quad u_2 = x^2(x-2)$ as vectors in U.

(b) Are u_1, u_2 linearly independent for $\mathbb{K} = \mathbb{R}$? **[5]**

(c) Do u_1, u_2 span U for $\mathbb{K} = \mathbb{R}$? **[5]**

(d) Repeating for $\mathbb{K} = \mathbb{F}_2$, the field of integers mod 2, are u_1, u_2 linearly independent in this case? Do they span U in this case? **[4]**

Justify all your answers.

Question 2 [20 marks].

(a) If U, W are subspaces of a vector space V, define the subspace $U + W$ and what is meant by $U \oplus W$. **[4]**

In the rest of this question, let $V = \mathbb{R}^3$ as column vectors and let

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V \mid x + y + 2z = 0,\ 2x - y - z = 0 \right\},$$

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V \mid 3x - y + z = 0 \right\}$$

be subspaces of V.

(b) What are the dimensions of U, W and what is $U \cap W$? **[4]**

(c) Is it true that $U + W = V$? **[4]**

(d) Is it true that $V = U \oplus W$? **[2]**

(e) Is it true that there exists a 3×3 matrix $\Pi \in M_3(\mathbb{R})$ such that $\Pi^2 = \Pi$ and $\text{ColumnSpace}(\Pi) = U$? **[6]**

(Hint: you may wish to consider a linear map $\pi : V \to V$ with matrix Π relative to the standard basis of V.)

Justify all your answers.

Question 3 [20 marks].

(a) The group S_2 consists of the identity and the transposition (12). Show that the Leibniz definition of the determinant reduces in the case of a 2×2 matrix to the usual Laplace formula. **[4]**

(b) Write $\begin{vmatrix} a+b & c \\ d+e & f \end{vmatrix}$ as the sum of two determinants of 2×2 matrices, for all $a, b, c, d, e, f \in \mathbb{K}$. **[4]**

(c) Working over $\mathbb{R}$, use row and column operations to put the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

into canonical form for equivalence. **[6]**

(d) Let V be a vector space over $\mathbb{R}$ with basis $v_1, \cdots, v_4$ and W a vector space with basis w_1, w_2. Let $\alpha : V \to W$ be the linear map with the matrix A in part (c) with respect to these bases. Find the nullity $\nu(\alpha)$ and determine image(α). **[6]**

Justify all your answers.

Question 4 [20 marks].

(a) Define the **minimal polynomial** $m_A(x)$ of an $n \times n$ matrix A over a field $\mathbb{K}$. **[4]**

(b) Find the **characteristic polynomial** $p_A(x)$ for

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix},$$

as a matrix over $\mathbb{C}$, and hence $m_A(x)$ in this case. Show your working. **[6]**

(c) Using the results of part (b), or otherwise, show that A **can** be diagonalised over $\mathbb{C}$ and find its eigenvalues. **[4]**

(d) Show that if the matrix in part (b) is regarded instead over $\mathbb{F}_2$ then it **cannot** be diagonalised. **[6]**

Question 5 [20 marks].

(a) Given an $n \times n$ real matrix $A = (a_{ij})$, define $q_A(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ as a quadratic form on $\mathbb{R}^n$. Why is it sufficient to take A here to be symmetric? [**4**]

(b) Let $q_A(x, y, z)$ on $\mathbb{R}^3$ be defined by

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

with x_i denoted as usual by x, y, z. By putting $q_A(x, y, z)$ into the canonical form in **Sylvester's Law of Inertia**, find the associated s, t such that A is congruent to the diagonal matrix with s entries 1, t entries -1 and zero elsewhere. [**7**]

(c) Why can we **not** use A in part (b) to define an inner product space $(\mathbb{R}^3, \cdot)$ with the standard basis v_i of $\mathbb{R}^3$ and $v_i \cdot v_j = a_{ij}$? [**3**]

(d) Give an example of an inner product space on $\mathbb{R}^3$ such that the associated matrix $A = (a_{ij})$ with respect to the standard basis is **not** a diagonal matrix. [**6**]

Justify all your answers.

End of Paper.