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B. Sc. Examination by Course Units

15 May 2014 Start: 2.30 pm

MTH6140 Linear Algebra II Duration: 2 hours

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Except for the matter of a mere pass, only the 4 best questions will be counted

CALCULATORS ARE NOT PERMITTED IN THIS EXAMINATION.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

IMPORTANT NOTE:
THE ACADEMIC REGULATIONS STATE THAT POSSESSION OF UNAUTHORISED MATERIAL AT ANY TIME WHEN A STUDENT IS UNDER EXAMINATION CONDITIONS IS AN ASSESSMENT OFFENCE AND CAN LEAD TO EXPULSION FROM QMUL.

PLEASE CHECK NOW TO ENSURE YOU DO NOT HAVE ANY NOTES, MOBILE PHONES OR UNATHORISED ELECTRONIC DEVICES ON YOUR PERSON. IF YOU HAVE ANY THEN PLEASE RAISE YOUR HAND AND GIVE THEM TO AN INVIGILATOR IMMEDIATELY. PLEASE BE AWARE THAT IF YOU ARE FOUND TO HAVE HIDDEN UNAUTHORISED MATERIAL ELSEWHERE, INCLUDING TOILETS AND CLOAKROOMS IT WILL BE TREATED AS BEING FOUND IN YOUR POSSESSION. UNAUTHORISED MATERIAL FOUND ON YOUR MOBILE PHONE OR OTHER ELECTRONIC DEVICE WILL BE CONSIDERED THE SAME AS BEING IN POSSESSION OF PAPER NOTES. MOBILE PHONES CAUSING A DISRUPTION IS ALSO AN ASSESSMENT OFFENCE.

EXAM PAPERS CANNOT BE REMOVED FROM THE EXAM ROOM.

Examiner: Prof. T. W. Müller

Question 1

Let V be a vector space over the field K.

(a) Explain what it means that V is *finitely generated.*

[3 marks]

(b) What does it mean for a set of vectors $b_1, b_2, \ldots, b_n \in V$ to be a *basis* of V? Provide two different definitions, and show their equivalence.

[6 marks]

(c) Define the *dimension* of a finitely generated vector space V. Explain briefly, why this concept is well defined.

[2 marks]

(d) Show that if V is finitely generated, then V has finite dimension.

[5 marks]

(e) Let V be finite-dimensional, and let $U \leq V$ be a subspace. Show that U is finite-dimensional, and that $\dim_K(U) \leq \dim_K(V)$.

[3 marks]

(f) Let

$$U = \left\{a_0 + a_1x + a_2x^2 + a_3x^3 : \ a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}$$

be the subset of the (real) vector space $V = \mathbb{R}[x]$ consisting of all polynomials of degree at most 3.

(i) Show that U is a subspace of V.

(ii) What is the dimension of U? Please justify your answer.

[6 marks]

Question 2

For Parts (a) and (b), let V and W be finitely generated vector spaces over a field K.

(a) (i) What does it mean to say that a map $T : V \to W$ is *linear*?

(ii) Show that the map $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 - 2x_2 + x_3$$

is linear.

[6 marks]

(b) Let $T : V \to W$ be a linear map.

(i) Define the *kernel* $\ker(T)$, the *image* $\operatorname{im}(T)$, the *rank* $\operatorname{rk}(T)$, and the *nullity* $\operatorname{nul}(T)$ of T.

(ii) State a result relating $\operatorname{rk}(T)$ to $\operatorname{nul}(T)$.

(iii) Does there exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^4$ such that $\operatorname{rk}(T) = \operatorname{nul}(T)$? Please justify your answer.

[6 marks]

(c) Let $S : U \to V$ and $T : V \to W$ be linear maps, where U, V, W are vector spaces over the same field K. Define the composition $T \circ S : U \to W$, and show that $T \circ S$ is again linear.

[5 marks]

(d) Let V be a vector space of dimension n over the field K, and let $\operatorname{End}_K(V)$ be the vector space consisting of all linear maps from V to V, with vector addition and scalar multiplication defined by

$$\begin{aligned} (S+T)(v) &:= S(v) + T(v), \quad (v \in V), \\ (\alpha S)(v) &:= \alpha S(v), \quad (\alpha \in K,\, v \in V), \end{aligned}$$

where $S, T \in \operatorname{End}_K(V)$.

(i) Exhibit an isomorphism $\operatorname{End}_K(V) \to M_n(K)$, where $M_n(K)$ is the K-vector space of n-dimensional square matrices over K (as introduced in the course). Verify your claim by explicitly describing the inverse isomorphism.

(ii) Use Part (i) to compute the dimension of $\operatorname{End}_K(V)$.

[8 marks]

Question 3

(a) (i) Define the determinant $\det(A) = |A|$ of an $n \times n$ matrix $A = (a_{i,j})_{1 \le i,j \le n}$ over a field K.

(ii) Show directly from the definition of a determinant that $|I_n| = 1$, where I_n denotes the n-dimensional identity matrix.

[5 marks]

(b) (i) Again arguing straight from the definition, compute the determinant of the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(ii) State Laplace's expansion theorem for an $n \times n$ matrix, and use it to reprove the result concerning $|B|$ obtained in Part (i).

[7 marks]

(c) Does there exist a linear map $T : \mathbb{C}^6 \to \mathbb{C}^3$ such that $(\mathrm{rk}(T))^2 = \mathrm{nul}(T)$? Please justify your answer.

[4 marks]

(d) (i) Define what it means for two matrices A, B to be *similar*.

(ii) Show that any two similar matrices have the same rank and determinant.

(iii) Are the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

similar? Please justify your answer.

(iv) The same question as in Part (iii) for the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 5 & 3 \\ -6 & -3 \end{bmatrix}.$$

[9 marks]

Question 4

(a) (i) Let A be an $n \times n$ matrix over a field K. Define the *characteristic polynomial* $c_A(x)$ of A. What is the degree of this polynomial (please justify your answer)?

(ii) Show that two similar square matrices A and B have the same characteristic polynomial.

(iii) Let $T : V \to V$ be a linear map on a vector space V. Define the characteristic polynomial $c_T(x)$ of T. Briefly explain, why this definition is unambiguous.

[10 marks]

(b) (i) State the Cayley–Hamilton Theorem.

(ii) Define the *minimal polynomial* $m_A(x)$ of a square matrix A, and show that $m_A(x)$ divides $c_A(x)$. What role does the Cayley–Hamilton Theorem play in this context?

(iii) Determine the minimal polynomial of the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

[10 marks]

(c) (i) Define what it means for a scalar $\lambda \in K$ to be an eigenvalue of a square matrix A over the field K.

(ii) With justification, list all those scalars $\lambda \in K$, which arise as eigenvalues of some projection $P : V \to V$, where $\dim_K(V) \geq 1$. Here, a linear map $P : V \to V$ is called a projection, if P satisfies the equation $P^2 = P$.

[5 marks]

Question 5

(a) Let V be a vector space over a field K.

(i) Given V and a positive integer m, define what is meant by a *linear m-form.* When is such a form called *alternating*?

(ii) Show that an alternating linear m-form changes sign when any two arguments are interchanged.

[8 marks]

(b) In the case where $K = \mathbb{R}$, define what is meant by an *inner product*, and an *inner product space.* Give an example of such a space.

[6 marks]

(c) (i) Define what it means for a basis $b_1, b_2, \ldots, b_n$ of an inner product space (V, b) to be *orthonormal.*

(ii) Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the inner product space (V, b), and let $v \in V$ be a vector with coordinate representation

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n.$$

Show that

$$\alpha_i = b(v, b_i), \quad 1 \leq i \leq n.$$

[5 marks]

(d) Let $V = \mathbb{R}^2$ be the 2-dimensional inner product space with inner product defined by

$$b\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) := x_1 y_1 + x_2 y_2.$$

Consider the basis $B = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ of V. Apply the Gram–Schmidt algorithm to transform B into an orthonormal basis of V.

[6 marks]

End of Paper