Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6132 Relativity

Duration: 2 hours

Date and time: 28 May 2014, 1430h-1630h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Dr. Juan A. Valiente Kroon

You are reminded of the following information, which you may use without proof.

- Lower case Latin indices run from 0 to 3.
- The metric tensor of the Minkowski spacetime is η_{ab} such that

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- The Lorentz transformations between two frames F and F' in standard configuration are given by

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z$$

where

$$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$

and F' is moving with speed v relative to F.

- The covariant derivative of a contravariant and a covariant vector are given, respectively, by

$$\nabla_a W^b = \partial_a W^b + \Gamma^b{}_{fa} W^f, \qquad \nabla_a V_b = \partial_a V_b - \Gamma^f{}_{ba} V_f.$$

- The metric tensor satisfies:

$$g_{ab} g^{bc} = \delta_a{}^c, \quad \nabla_a g_{bc} = 0.$$

- Christoffel symbols (connection):

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

- The Riemann curvature tensor:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$

- Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

- Geodesic equations:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

SECTION A: *You should attempt all questions. Marks awarded are shown next to the questions.*

Question 1

(i) Explain what is understood by an inertial system of reference.

[3 marks]

(ii) Let F and F' denote two inertial reference systems moving with velocity v with respect to each other along the x-axis. On page 2 you are given the Lorentz transformation expressing the coordinates (t', x', y', z') in terms of the coordinates (t, x, y, z). Give the inverse transformation —that is, the transformation expressing the coordinates (t, x, y, z) in terms of the (t', x', y', z') coordinates.

[3 marks]

(ii) Consider a particle moving along the x-axis. Its velocity in the x direction with respect to the frames F and F' is given, respectively, by

$$V = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad V' = \frac{\mathrm{d}x'}{\mathrm{d}t'}$$

Use the Lorentz transformation between two frames F and F' in standard configuration to show that

$$V' = \frac{V - v}{1 - Vv/c^2}.$$

[4 marks]

Question 2

(i) Define what is meant by the scalar product of two arbitrary 4-vectors $\bar{A}$ and $\bar{B}$.

[3 marks]

(ii) Write down the conditions for a 4-vector to be timelike, spacelike and null, respectively.

[3 marks]

(iii) Consider the 4-vectors

$$\bar{A} = (a, 0, 2a, 0), \qquad \bar{B} = (2b, 0, b, 0)$$

where a and b are arbitrary constants. Determine whether $\bar{A}$ and $\bar{B}$ are timelike, spacelike or null. Under which conditions on a and b would $\bar{C} = \bar{A} + \bar{B}$ be null?

[4 marks]

Question 3

(i) Give the transformation rule of a $(2,2)$-tensor $T_{ab}{}^{cd}$. Show that $T_{ab}{}^{ad}$ is a $(1,1)$-tensor.

[5 marks]

(ii) Give the definition of the Kronecker's delta $\delta_a{}^b$. Show that $\delta_a{}^b$ is a $(1,1)$-tensor.

[5 marks]

Question 4
Let F and F' denote two inertial reference systems moving with velocity v with respect to each other along the x-axis.

(i) If $\Delta t'$ denotes an interval of time as measured by a clock at rest with respect to F', use the Lorentz transformations given in page 2 to find the interval of time Δt as measured by F.

[3 marks]

(ii) If $\Delta x'$ denotes the length of a rod moving along the x-axis with velocity v, find its length Δx as measured by F.

[3 marks]

(iii) Draw a 2-dimensional spacetime diagram of the following situation: two mirrors are located, respectively at $x = -x_0$ and $x = x_0$; two rays of light are shot from the origin at $t = 0$, one towards the right one towards the left. To draw the diagram use units for which $c = 1$.

[4 marks]

Question 5 The metric for a particular 2-dimensional spacetime is given by

$$\mathrm{d}s^2 = -e^{2Ar}\mathrm{d}t^2 + \mathrm{d}r^2$$

where A is an arbitrary constant. Let $(x^1, x^2) = (t, r)$.

(i) Compute, by the method you prefer all the Christoffel symbols $\Gamma^a{}_{bc}$.

[7 marks]

(ii) Using the formula

$$R_{ab} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ca} - \Gamma^c{}_{da}\Gamma^d{}_{cb} + \Gamma^c{}_{cd}\Gamma^d{}_{ab}$$

compute the R_{11} component of the Ricci tensor.

[3 marks]

SECTION B: *Each question carries 25 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.*

Question 6 In this question consider units for which $c = 1$.

(i) What is the 4-momentum, $\bar{p}$, of a photon of frequency ν moving along the positive direction of the x-axis of an inertial system of reference F? Show that for a photon one always has that $|\bar{p}|^2 = 0$. What is the meaning of this result?

[5 marks]

(ii) Consider a further inertial system of reference F' moving with velocity v along the x-axis of F. If $\bar{p}'$ denotes the 4-momentum of the photon with respect to F', give the relation between the components of $\bar{p}$ and $\bar{p}'$. Use this result to show that

$$\frac{\nu'}{\nu} = \sqrt{\frac{1-v}{1+v}},$$

where ν' is the frequency of the photon as measured by F'.

[8 marks]

(iii) A particle of rest mass m_0 moving along the x-axis with speed v decays into two particles, each with a rest mass $m_0/2$. Both particles continue to move along the x-axis. Show that the new particles move with the same speed.

[12 marks]

Question 7

(i) Let $W_a{}^b$ denote a $(1,1)$-tensor. Give the formula for $\nabla_c W_a{}^b$ in terms of ∂_c and the Christoffel symbols $\Gamma^a{}_{bc}$.

[4 marks]

(ii) Show that $\nabla_c \delta_a{}^b = 0$.

[4 marks]

(iii) What is the physical/geometrical meaning of the metric tensor g_{ab}?

[3 marks]

(iv) Let g_{ab} denote a metric tensor. Using $\nabla_c g_{ab} = 0$ and the result of (ii), show that

$$\nabla_c g^{ab} = 0$$

where g^{ab} denotes the contravariant metric.

[7 marks]

(v) Let $x^a(\lambda)$ with λ an affine parameter, denote a curve in spacetime with tangent vector $v^a = \mathrm{d}x^a/\mathrm{d}\lambda$. Use the formulae for the covariant derivative of a contravariant vector to show that the geodesic equation equation

$$v^b \nabla_b v^a = 0$$

can be rewritten as

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\lambda^2} + \Gamma^a{}_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \frac{\mathrm{d}x^c}{\mathrm{d}\lambda} = 0.$$

[7 marks]

Question 8

The Riemann curvature tensor of a certain spacetime is of the form

$$R_{abcd} = K \left(g_{ac} g_{bd} - g_{ad} g_{cb} \right),$$

with K a constant.

(i) Show that the tensor R_{abcd} satisfies the identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0.$$

[6 marks]

(ii) Show that

$$\nabla_e R_{abcd} = 0$$

[4 marks]

(iii) Show that the corresponding Ricci tensor R_{ab} is proportional to the metric, and that the Ricci scalar R is a constant. [7 marks]

(iv) Show that for the above curvature tensors one has that

$$\nabla^a R_{ab} - \frac{1}{2} \nabla_b R = 0.$$

[3 marks]

(v) Under which conditions is a metric g_{ab} with the above curvature tensor a solution to the vacuum Einstein field equations? Which spacetime satisfies this condition?

[5 marks]

End of Paper