Queen Mary
University of London

Main Examination period 2019

MTH6128 / MTH6128P: Number Theory

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: S. Lester, A. Saha

Question 1. [20 marks]

(a) Define the terms **algebraic number** and **minimal polynomial**. State the **Chinese Remainder Theorem**. **[6]**

(b) Give an example of an algebraic integer, which is not an integer. Explain why the number you gave has the desired properties. **[3]**

(c) Find all integer solutions to the system of congruences

$$x \equiv 1 \pmod 7$$
$$x \equiv 2 \pmod{30}.$$

Explain your working. **[6]**

(d) Determine the minimal polynomial of $\frac{\sqrt{7}}{2} - \frac{9}{2}$. **[5]**

Question 2. [15 marks]

(a) Find the value of the continued fraction

$$[4; \overline{1, 6}].$$

Your answer should be a number of the form $u + v\sqrt{d}$, where $u, v \in \mathbb{Q}$, $d \in \mathbb{N}$. **[5]**

(b) Let x be an irrational number and n be a positive integer. Let $c_n = p_n/q_n$ be the nth convergent of the continued fraction of x.

(i) Prove that **[5]**

$$\frac{1}{q_n q_{n+1}} = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \left| x - \frac{p_{n+1}}{q_{n+1}} \right| + \left| x - \frac{p_n}{q_n} \right|.$$

State precisely all results from the lectures you use in the proof.

(ii) Prove that $\frac{1}{q_n q_{n+1}} < \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$. **[2]**

(iii) Use parts (i) and (ii) to prove that **[3]**

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{2q_n^2} \qquad \text{or} \qquad \left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \frac{1}{2q_{n+1}^2}.$$

Question 3. [15 marks]

(a) Given that

$$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}],$$

find the fundamental solution to

$$x^2 - 19y^2 = \pm 1.$$

Use your answer to write down all positive integer solutions to the equation $x^2 - 19y^2 = 1$. Explain why you have found ALL solutions. **[9]**

(b) Given that $25^2 \equiv -1 \pmod{313}$ use Hermite's algorithm to find integers x, y such that

$$x^2 + y^2 = 313.$$ **[6]**

Question 4. [13 marks]

(a) Define **Euler's ϕ-function**. Define the term **primitive root** $\pmod{p}$, where p is prime. **[4]**

(b) Find a primitive root $\pmod{29}$. Explain why the integer you gave has the desired properties. **[5]**

(c) Find the number of primitive roots $\pmod{101}$. Explain your working. **[4]**

Question 5. [25 marks]

(a) Define the term **quadratic residue**. State **Euler's Criterion.** **[5]**

(b) For each of the equations, find all integers strictly between 0 and 53 which are solutions to the following equations. Use the methods developed in the lectures to solve the equation $x^2 \equiv a \pmod{p}$ and explain your working.

(i) $x^2 \equiv 35 \pmod{53}$ **[6]**

(ii) $x^2 \equiv -1 \pmod{53}$ **[6]**

(c) Prove there are infinitely many prime numbers congruent to 1 $\pmod{4}$. **[8]**

Question 6. [12 marks]

(a) State **Hensel's Lemma**. **[3]**

(b) Use Hensel's Lemma to find all integer solutions to the equation

$$x^2 - 5 \equiv 0 \pmod{19^2}.$$

Explain your working. **[9]**

End of Paper.