Queen Mary
University of London

MTH6127 / MTH6127P: Metric Spaces and Topology

Duration: 2 hours

Date and time: 12th May 2016, at 14.30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

This paper has two sections.
You should attempt all the questions in Section A. In Section B you may attempt as many questions as you wish. Except for the award of a bare pass, only the best TWO questions answered in Section B will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): M. Farber

In this examination $\mathbb{R}$ stands for the set of real numbers, $\mathbb{Q}$ stands for the set of rational numbers and $\mathbb{Z}$ stands for the set of integers.

Section A: Each question carries 10 marks.

Question 1.

(a) Give the definition of *a metric space* (X, d). **[2]**

(b) Explain what it means for a subset $U \subseteq X$ in a metric space to be *open*. **[2]**

(c) Show that any open set $U \subset X$ in a metric space is a union of a family of open balls $B(c, r)$ having radii $r < 0.01$. **[4]**

(d) Can we replace the number 0.01 by 0.001 in part (c)? **[2]**

Question 2.

(a) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a metric space X *converges* to a point $x_0 \in X$? **[2]**

(b) When do we say that a sequence $\{x_n\}_{n\geqslant 1}$ of points in a topological space X *converges* to a point $x_0 \in X$? **[2]**

(c) Let X be a metric space. Is it possible that a sequence of points $\{x_n\}_{n\geqslant 1}$, $x_n \in X$ converges to two distinct points $x_0, x_0' \in X$, $x_0 \neq x_0'$? Justify your answer. **[2]**

(d) Consider $X = \mathbb{R}$ with the finite-complement topology (i.e. when open subsets are complements of the finite subsets). Consider the sequence $x_n = n \in X$ and find all points $x_0 \in X$ such that the sequence $\{x_n\}_{n\geqslant 1}$ converges to x_0. Justify your answer. **[4]**

Question 3.

(a) Explain what is meant for a metric space (X, d) to be *complete* and give an example of a metric space which is not complete. Justify your answer. **[2]**

(b) Let X be a metric space and let $A \subseteq X$ be a subset which is not closed. Show that A is not complete with respect to the induced metric. **[2]**

(c) Which of the following subsets of $\mathbb{R}$ are complete when considered as subspaces of $\mathbb{R}$ with the usual metric? Briefly justify your answer.

(i) $\{2^n;\, n = 1, 2, \dots\}$, **[2]**

(ii) $\{2^{-n};\, n = 1, 2, \dots\}$, **[2]**

(iii) $\{2^{-n};\, n = 1, 2, \dots\} \cup \{0\}$. **[2]**

Question 4.

(a) Give the definition of *a topological space*. **[3]**

(b) When do we say that a topological space is *Hausdorff*? **[2]**

(c) Give an example of a topological space which is not Hausdorff. **[3]**

(d) Show that any metric space with the topology induced by the metric is Hausdorff. **[2]**

Section B: Each question carries 30 marks.

Question 5.

(a) When do we say that a map $f : X \to Y$ between topological spaces is *continuous*? **[3]**

(b) When do we say that a subset of a topological space is *closed*? **[3]**

(c) Show that a map $f : X \to Y$ between topological spaces is continuous if the inverse image $f^{-1}(F) \subseteq X$ of any closed set $F \subset Y$ is closed. **[4]**

(d) Let $f : X \to Y$ be a continuous map between topological spaces. Assume that $\{x_n\}_{n \geqslant 1}$ is a sequence of points $x_n \in X$ which converges to $x_0 \in X$. Prove that the sequence $\{f(x_n)\}_{n \geqslant 1}$ converges to $f(x_0)$ in Y. **[8]**

(e) When do we say that two topological spaces are *homeomorphic*? **[4]**

(f) Give an example of two homeomorphic metric spaces X and Y such that X is complete and Y is not complete. Justify your answer. **[8]**

Question 6.

(a) What is meant by *an open cover* of a topological space? **[2]**

(b) When do we say that a topological space is *compact*? **[3]**

(c) Which of the following subsets of the real line $\mathbb{R}$ are compact? Briefly justify your answer:

(i) $[2, 3]$; **[3]**

(ii) $(2, 3)$; **[3]**

(iii) $[2, \infty)$; **[3]**

(iv) $\mathbb{R}$; **[3]**

(v) $\{n^{-3}; n = 1, 2, \dots\}$. **[3]**

(d) Prove that any compact subset of a Hausdorff topological space is closed. **[10]**

Question 7.

(a) Let (X, d) be a metric space. When do we say that a mapping $f : X \to X$ is *a contraction*? **[4]**

(b) State the contraction mapping theorem. **[5]**

(c) Consider $\mathbb{R}^2$ with the d_1-metric, i.e. $d_1(v, v') = |x - x'| + |y - y'|$ where $v = (x, y)$ and $v' = (x', y')$. Is this metric space complete? Justify your answer. **[5]**

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(v) = (\frac{1}{3}y, \frac{1}{3}(x + 1))$, where $v = (x, y)$. Show that f is a contraction with respect to the d_1-metric. **[10]**

(e) Find the fixed point of f. **[6]**

End of Paper.