Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6126 Metric Spaces

Duration: 2 hours

Date and time: 9th May 2014, 14.30–16.30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Candidates should note that the Examination and Assessment Regulations state that possession of unauthorized materials by any candidate who is under examination conditions is an assessment offence. Please check your pockets now for any notes that you may have forgotten that are in your possession. If you have any, then please raise your hand and give them to an invigilator now.

Exam papers must not be removed from the examination room.

Examiner(s): David Arrowsmith

In this examination, $\mathbb{N} = \{1, 2, 3, \ldots\}$ stands for the set of natural numbers, $\mathbb{Z}$ stands for the set of integers, $\mathbb{Q}$ stands for the set of rational numbers, and $\mathbb{R}$ stands for the set of real numbers.

Section A: Each question carries 10 marks. You should attempt ALL FOUR questions.

Question 1

Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function.

(a) State the three axioms that d must satisfy in order to be a metric on X. [2]

(b) Let d be a metric on X and define $\sigma : X \times X \to \mathbb{R}$ by $\sigma(x, y) = +\sqrt{d(x, y)}$ for all $x, y \in X$. Prove that (X, σ) is a metric space. [4]

(c) Let X be a set, and $a > 0$ a fixed real constant. Prove that the function $d : X \times X \to \mathbb{R}$.

$$d(x, y) = \begin{cases} a & \text{if } x \neq y; \\ 0 & \text{if } x = y, \end{cases}$$

is a metric on X. [4]

Question 2

Let (X, d) be a metric space. Suppose that $A \subseteq X$.

(a) What does it mean to say a point $x \in X$ is **not** an accumulation point of A? [3]

(b) Suppose that A and B are subsets of X and suppose that x is not an accumulation point of A and not an accumulation point of B. Working directly from the definition, prove that x is not an accumulation point of $A \cup B$. [7]

Question 3

(a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Give two (equivalent but different) definitions for f to be continuous. [4]

(b) Consider the metric spaces $(\mathbb{R}^3, d_1)$ and $(\mathbb{R}, d_1)$, where d_1 is the Manhattan metric in each case. Using one of the definitions from (a), prove that the function $f : \mathbb{R}^3 \to \mathbb{R}$ where $f(x, y, z) = x + 3y - z$ is continuous. [6]

Question 4

(a) Let (f_n) be a sequence of functions between two metric spaces (X, d_X) and (Y, d_Y). Define what it means for the sequence (f_n) to converge *pointwise* to a function $f : X \to Y$. [2]

(b) For each of the following sequences (f_n) of functions in $\mathcal{C}[0, \infty)$, the metric space of real-valued continuous functions $f : [0, \infty) \to \mathbb{R}$ with the *sup* metric, decide whether the sequence converges pointwise to a function f in either $\mathcal{C}[0, \infty)$, or in the corresponding space of bounded functions $\mathcal{B}[0, \infty)$. If the sequence f_n converges pointwise to f, determine whether the sequence converges to f uniformly.

(i) $f_n(x) = e^{-nx^2}$. [4]

(ii) $f_n(x) = xe^{-nx^2}$. [4]

State any theorems that you use.

Section B: Each question carries 30 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 5

(a) What is meant by $x \in U$ being an interior point of $U \subseteq X$ in the metric space (X, d)? Prove that U is open iff $int(U) = U$. [8]

(b) Suppose that X is a set with a metric given by

$$d_X(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \neq x_2; \\ 0 & \text{if } x_1 = x_2. \end{cases}$$

Let also (Y, d_Y) be an arbitrary metric space. Show that every singleton set $\{x\} \subseteq X$ is an open set of (X, d). Hence, or otherwise, show that every subset of X is an open set, stating clearly any results that you use. Explain why every function $f : X \to Y$ is continuous, stating the definitions and properties needed. [8]

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ x + 1 & \text{if } x > 0. \end{cases}$$

Describe the inverse image $S = f^{-1}((-\infty, 1/2))$. Is the set S open in $\mathbb{R}$ with the Euclidean metric? What does your answer imply about the continuity of f? [8]

(d) What does it mean to say that a set $A \subseteq X$ in the metric space (X, d) is path connected? Show that the subset $A = \{\, (x, \sin(1/x)) \,|\, x \in \mathbb{R}^+\}$ of $\mathbb{R}^2$ is path-connected. [6]

Question 6

Let $\mathcal{C}$ be the set of all continuous real-valued functions on the interval $[0, \frac{1}{2}]$.

(a) Give the definition of the uniform metric d (also called the *sup* metric) on $\mathcal{C}$. Explain why the sup metric can be replaced by the max metric. [8]

(b) Let $T : \mathcal{C} \to \mathcal{C}$, be defined by $f \mapsto Tf : [0, \frac{1}{2}] \to \mathbb{R}$ where

$$(Tf)(t) = t(f(t) + 1), \text{for all}\, t \in [0, \frac{1}{2}].$$

Show that $T : \mathcal{C} \to \mathcal{C}$ defined by $f \mapsto Tf$, for all $f \in \mathcal{C}$, is a contraction mapping on $(\mathcal{C}, d)$ by considering $d(Tf, Tg)$, $f, g \in \mathcal{C}$. [10]

(c) What property of the space (C, d) ensures that the Contraction Mapping Theorem can be applied, given that T is a contraction? (You are not asked to show that $\mathcal{C}$ has this property.) [6]

(d) In view of Part (d), the Contraction Mapping Theorem can indeed be applied to deduce that there is a unique $f \in C$ such that $Tf(t) = f(t)$ for $t \in [0, \frac{1}{2}]$. What is this f? [6]

Question 7

(a) Explain what it means for a subset K of a metric space (X, d) to be compact. [6]

(b) From first principles — i.e., directly from the definition you gave in part (a) — prove that the following subsets of $\mathbb{R}$ are not compact (with the usual metric).

(i) $[0, \infty)$, [3]

(ii) $[0, 1)$. [3]

(c) Prove that any closed subset of a compact metric space is compact. [8]

(d) Let K, L be compact subsets of a metric space. Prove that $K \cup L$ is compact. [10]

End of Paper