Queen Mary
University of London

MTH6121: Introduction to Mathematical Finance

Duration: 2 hours

Date and time: 18th May 2016, 10:00-12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): N. Rodosthenous and O. Bandtlow

Question 1.

(a) Suppose that the monthly compounded annual interest rate is 2.5%. What is the present value β of 1 pound received in 1 month from today (in other words, the discounting factor)? Give your answer to 6 decimal places. [**2**]

(b) Using the rate in part (a), suppose that you buy a television for 2000 pounds. The store gives you a deal for which you pay immediately 250 pounds and then make a payment A at the end of each month for 5 years until your debt is paid off. What is the payment A? [**6**]

(c) You are offered two different jobs. The salary paid at the end of each year (in thousands of pounds) is

Job	1	2	3	4	5
A	12	14	16	18	20
B	16	16	15	15	15

Which job pays best if the annual nominal interest rate is $r = 0.1$? Which job pays best if instead the annual nominal interest rate is $r = 0.2$? [**4**]

(d) If a bank offers a continuously compounded interest at nominal rate 2%, how long does it take for the amount of money in the bank to triple? [**3**]

(e) Suppose now that interest rate is time-varying and given by $r(t) = 0.02\,(1 + t^{0.1})$. Determine the yield curve. [**4**]

(f) Suppose that you rent a flat in London for which you have to pay a deposit of 2000 pounds. In order to pay this you get a loan from a bank offering a time-varying interest rate modelled as in part (e). In one year's time you pay back the bank 1000 pounds. How much do you then still owe the bank in one year's time? [**3**]

(g) In the setting of part (f), what is the landlord's internal rate of return in three years' time when he is required by the contract to pay you back the deposit in full (without interest)? [**3**]

Question 2.

(a) What is an arbitrage opportunity? Give two types of such opportunities. [**4**]

(b) What are the main differences between a European option and a Forward contract? What kind of standard European option would you choose to hedge the risk of an underlying asset price increase? [**4**]

(c) Suppose that an asset has spot price S at time $t = 0$ and there exists a continuously compounded interest rate r. What is the no-arbitrage forward price F agreed at $t = 0$ for the delivery of the underlying asset at $t = T$? Give a detailed explanation for your answer.
Hint: The above description defines a forward contract. [**6**]

(d) Assume that the interest rate is 2% per time period. Suppose that at time 0, company A's share is traded at the price $£85$, and that at the next time period 1, your financial analyst predicts that it will be traded either at $£81$ or $£88$. Consider a European call option on this share with maturity time period 1 and strike price $£86$. What is the no-arbitrage price C of this call option?
Hint: Use the Arbitrage Theorem with two wagers; purchase one share and purchase one call option. [**7**]

(e) What is the condition that needs to be satisfied by upward and downward movements of the underlying asset price, which guarantees that there is no arbitrage in the market? Is this condition satisfied in the setting of part (d)? [**4**]

Question 3.

(a) What is a stochastic process? [**2**]

(b) What is a Markov process? [**2**]

(c) An example of a Markov process is a Brownian motion with drift, denoted by $Y(t)$, with drift parameter $\mu = 0.25$ and volatility parameter $\sigma = 0.5$. What distribution does $Y(t)$ follow? [**2**]

(d) What is the drawback of a Brownian motion with drift that may rule it out as a suitable model for share prices? [**2**]

(e) Suppose now that $S(t)$ denotes the annual share price of company B at time t and its evolution is given by the Black and Scholes model. Write down the expression for $S(t)$. [**3**]

(f) What is the "Black and Scholes" price C of a European call option with maturity $T = 3$ months and strike price $K = £11$, written on company B's share with current price $S = £10$ and volatility $\sigma = 0.25$? Suppose that the (continuously compounded) interest rate offered by the market is $r = 1.5\%$. [**5**]

(g) What is the price P of a European put option on company B's share with the same maturity $T = 3$ months and strike price $K = 11$ pounds?
Hint: You may use the Put-Call parity formula. [**3**]

(h) What is the rate with which the call option price C in part (f) changes if the current share price S is varied and the rest of the model parameters σ, K, r, T are kept constant? No proof is required.
Hint: The above description defines a partial derivative of C. [**3**]

(i) Suppose that the underlying share of company B does not pay any dividends between time 0 and maturity time T. How would the price C of the call option in part (f) change, if the option was of American type instead of European? [**3**]

Question 4.

(a) Explain what is meant by a lognormally distributed random variable Y with parameters μ and σ. What values can the parameters take? [**3**]

(b) Show that the p.d.f. of a lognormal random variable $Y \sim \mathrm{LogNormal}(\mu, \sigma^2)$ is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} \exp\left(-\frac{(\log y - \mu)^2}{2\sigma^2}\right) & \text{if } y > 0; \\ 0 & \text{if } y \leq 0. \end{cases}$$

Hint: You may use the theorem on the transformation of probability density functions of continuous random variables. [**5**]

(c) Suppose that $S(n)$ denotes the price of a financial product at the end of the n-th month, where $n \in \mathbb{N}$. What does it mean to say that $S(n)$ is given by the IID lognormal model? [**3**]

(d) Assume that the parameters of the IID lognormal model are $\mu = 0.0165$ and $\sigma = 0.0730$. Show that the probability that the price increases over the first month is 59.10%. [**5**]

(e) What is the probability that the price keeps increasing in each of the first two months? [**3**]
Hint: You may use part (d).

(f) What is the probability that the price is higher at the end of the second month than at the starting time 0? Show all steps and prove all statements used. [**6**]

End of Paper—An appendix of 1 page follows.

Table of the cumulative standard normal distribution

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

End of Appendix.