Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH6120: Further Topics in Mathematical Finance

Duration: 2 hours

Date and time: 6th May 2015, 14:30–16:30

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): S. del Bano Rollin

 TURN OVER

WRITE CLEARLY AND LEGIBLY

Question 1 (25 marks).

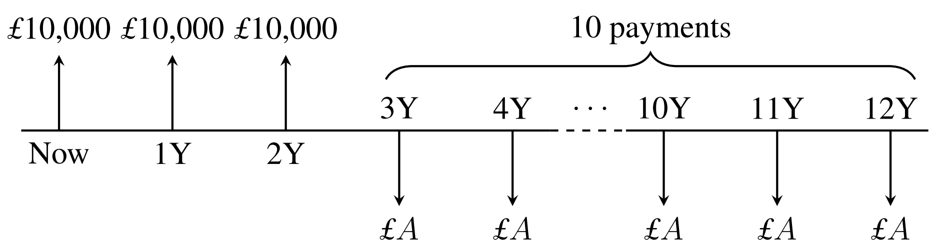
(a) Explain what the *effective rate* or *Annual Percentage Rate* (APR) is and why is it used. **[5]**

(b) Calculate the APR corresponding to an annualised semi-annual interest rate r. Is the APR larger or smaller that r? **[5]**

(c) We place £1,000 in a deposit which pays a 5% continuous interest rate. How long will it take for the deposit to grow to £1,500?

Hint: You may use the approximation $\ln(x) \sim (x-1) - \frac{1}{2}(x-1)^2$. **[5]**

(d) Imagine you require to receive three payments of £10,000: one now, one in one year and one in two years. You want to fund this with payments of an amount A in the following 10 years. Schematically:



Assuming the one year interest rate is r what should A be?

Hint: Check page 5 for calculation of geometric sums. **[5]**

(e) Find A if there are infinitely many payments made instead of just 10 in section (d) (you may wish to assume $r > 0$ if needed).

Calculate A in the case $r = 1\%$.

Hint: Check page 5 for calculation of geometric series. **[5]**

Question 2 (25 marks).

(a) Explain what *Geometric Brownian Motion* is. **[5]**

(b) Why are financial assets better modelled by Geometric Brownian Motion than by Brownian Motion? **[5]**

(c) Let $a, b \in \mathbb{R}$. Show that if $X \sim \mathcal{N}(0,1)$ then

$$\mathbb{E}\left(\mathrm{e}^{a+bX}\right) = \mathrm{e}^{a+b^2/2}.$$

Hint: Write the expectation above as an integral involving the pdf of X, and use *completion of squares*: $a + bx - \frac{x^2}{2} = a + \frac{b^2}{2} - \frac{(x-b)^2}{2}$. Check page 6 for formula of density of a normal distribution function. **[5]**

(d) Derive from the expression above the expectation of Geometric Brownian Motion. **[5]**

(e) What does *Risk Neutral* Geometric Brownian Motion mean? **[5]**

Question 3 (25 marks). A *participating forward* with expiry T and strike K is a contract whereby:

- You have to buy A units of the asset if the asset value at expiry is above K.
- You have to buy B units of the asset if the asset value at expiry is below K.

Assume A, $B > 0$ for simplicity.

(a) Show that the payoff is:

$$\text{Payoff} = \begin{cases} A(S-K) & \text{if } S \le K \\ B(S-K) & \text{if } S > K, \end{cases}$$

and plot a graph of this function. **[6]**

(b) Show that a participating forward is equal to a combination of calls and puts. Discuss how to price a participating forward.

Hint: Show that the payoff can also be written as $\alpha\,(S-K)^+ + \beta\,(K-S)^+$ for certain α and β. **[6]**

(c) Calculate the price of a participating forward in the case K is equal to the forward rate, F. Simplify the result as much as possible.

Hint: See page 5 for the Black–Scholes pricing formulæ. **[7]**

(d) Explain what put-call parity is and how it is proved. **[6]**

Question 4 (25 marks).

(a) Explain what *log-return* means and how it relates to the usual return. [**5**]

(b) Explain what a *utility function* is. [**5**]

(c) Explain what a *risk averse* utility function is and provide an example. [**5**]

(d) An investor is considering placing a fraction, α, of their capital in a risky investment that either results in their investment doubling with probability p or in it being wiped out with probability $q = 1 - p$. The rest of the capital is left uninvested. We assume interest rates are zero.

If the risk preferences of the investor are defined by the utility function $u(x) = 1 - \exp(-\beta x)$, what fraction of their capital should they invest? [**10**]

End of Paper—An appendix of 2 pages follows.

The Black Scholes Formula

The price of a *European call option* with strike K and years to expiry T is

$$\begin{aligned} Call &= \mathrm{e}^{-rT}\left(FN(d_1) - KN(d_2)\right) \\ d_1 &= \frac{\ln(F/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2\sqrt{T} \\ d_2 &= \frac{\ln(F/K)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma^2\sqrt{T} \end{aligned}$$

where $F = \mathrm{e}^{rT}S$ is the forward rate, r the interest rate, S the current value of the asset, and σ its volatility.
$N(x)$ is the standard normal cumulative distribution function.
The price of a *European put option* with strike K and years to expiry T is

$$Put = \mathrm{e}^{-rT}\left(-FN(-d_1) + KN(-d_2)\right)$$

where F, d_1 and d_2 are defined as above.

The Black Scholes Formula (last year's version)

The price of a *European call option* with strike K and years to expiry T is

$$\begin{aligned} C &= S_0\phi(\omega) - K\mathrm{e}^{-rT}\phi(\omega - \sigma\sqrt{T}) \\ \omega &= \frac{rT + \frac{1}{2}\sigma^2 t - \log\frac{K}{S_0}}{\sigma\sqrt{T}} \end{aligned}$$

where S_0 is the current share price, r the interest rate, and σ its volatility.
$\phi(x)$ is the standard normal cumulative distribution function.
The price of a *European put option* with strike K and years to expiry T is

$$P = -S_0\phi(-\omega) + K\mathrm{e}^{-rT}\phi(-\omega + \sigma\sqrt{T})$$

where ω is defined as above.

Geometric sum

For integers $a \leq b$, we have:

$$x^a + x^{a+1} + \cdots + x^b = \frac{x^a - x^{b+1}}{1-x}.$$

If $|x| < 1$, then taking $\lim_{b\to\infty}$ in the result above yields the geometric series:

$$x^a + x^{a+1} + \cdots = \sum_{n=a}^{\infty} x^n = \frac{x^a}{1-x}.$$

Normal density function

The probability density function of a standard normal random variable, $X \sim \mathcal{N}(0,1)$, is:

$$pdf_X(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}.$$

End of Appendix.