Queen Mary University of London

Main Examination period 2019

MTH6109 / MTH6109P: Combinatorics

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Question 1. [17 marks]

(a) State the **sum rule** and the **correspondence principle** of combinatorics. **[4]**

(b) For integers n and k with $0 \leqslant k \leqslant n$ the binomial coefficient $\binom{n}{k}$ is defined to be the number of k-element subsets of an n-set. **Using this definition**, show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{for } 1 \leqslant k \leqslant n.$$ **[6]**

(c) Making use of the recurrence relation in part (b), prove that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{for } 0 \leqslant k \leqslant n$$

where, for a non-negative integer n, we define

$$n! = \begin{cases} \prod_{i=1}^{n} i & \text{if } n > 0 \\ 1 & \text{if } n = 0. \end{cases}$$ **[7]**

Question 2. [28 marks]

(a) State and prove the binomial theorem for non-negative integral exponents. **[6]**

(b) Provide a bijective proof of the following statement: the number of even-size subsets of a non-empty finite set S equals the number of odd-size subsets. **[6]**

(c) Define the **Stirling numbers** $S(n,k)$ of the second kind, as well as the **Bell numbers** $b(n)$. What is the connection between $b(n)$ and the Stirling numbers $S(n,k)$? **[5]**

(d) Straight from the definition, compute the numbers $S(n,2)$ for $n \geqslant 0$. **[4]**

(e) (i) Define the **multinomial coefficient**

$$\binom{n}{n_1, \ldots, n_k},$$

and explain the counting problem to which it is the answer. **[3]**

(ii) Let $a = (a_1, a_2, a_3)$ be a triple of non-negative integers, and let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ be the coordinate vectors. Show that the number of lattice paths in the 3-dimensional integral grid from the origin $(0,0,0)$ to the point a using steps e_1, e_2, e_3 is given by the multinomial coefficient

$$\binom{a_1 + a_2 + a_3}{a_1, a_2, a_3}.$$ **[4]**

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Question 3. [16 marks]

(a) Define what is meant by a **simple graph** on a vertex set Ω. How many simple graphs are there on a given set of n labelled vertices? (You need not justify your answer.) **[4]**

(b) Define what is meant by an **isomorphism** $\phi : G \to G'$, where $G = (V, E)$ and $G' = (V', E')$ are graphs. **[3]**

(c) Consider the graphs $G = (V, E)$ and $G' = (V', E')$ with vertex sets $V = V' = [5]$, and with edge sets

$$E := \big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\big\},$$
$$E' := \big\{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,5\}\big\}.$$

Write down an isomorphism from G to G'. **[4]**

(d) Prove that every simple graph on $n \geqslant 2$ vertices contains vertices v_1, v_2 with $v_1 \neq v_2$ and $\deg(v_1) = \deg(v_2)$.

[Hint: what are the possible values of $\deg(v)$? Apply the pigeonhole principle.] **[5]**

Question 4. [13 marks]

(a) Define the **generalized binomial coefficient** $\binom{\alpha}{k}$, where α is any complex number and k is a non-negative integer. Show that

$$\binom{-n}{k} = (-1)^k \binom{n-k+1}{k},$$

where n and k are non-negative integers. **[6]**

(b) (i) State the **inclusion/exclusion principle** for n finite sets $A_1, A_2, \ldots, A_n$. **[3]**

(ii) Suppose you are given the following information concerning the three finite sets A, B, and C:

$$|A| = 7, \qquad |A \cap B| = 3,$$
$$|B| = 8, \qquad |A \cap C| = 2,$$
$$|C| = 5, \qquad |B \cap C| = 3.$$

Which values can $|A \cup B \cup C|$ take? Justify your answer. **[4]**

Question 5. [26 marks]

(a) Let $G = (V, E)$ be a (simple) graph. Define the concepts of a **walk**, **path**, and **cycle** in G. When is G called **connected**? **[5]**

(b) When is a graph G called a **tree**? **[2]**

(c) Suppose that $T = (V, E)$ is a finite tree with $|V| \geqslant 2$.

(i) Show that T has (at least) two leaves (i.e. vertices of degree 1). **[4]**

(ii) State the induction lemma for finite trees. **[3]**

(iii) Making use of part (ii), show that a finite tree with n vertices has precisely $n - 1$ edges. **[3]**

(d) Let $G = (V, E)$ be a graph.

(i) When is G called **bipartite**? **[2]**

(ii) State an alternative characterization of bipartite graphs in terms of the lengths of cycles. **[3]**

(iii) Consider the graph G with $V = [6]$ and

$$E = \{\{1,2\}, \{1,4\}, \{1,6\}, \{2,3\}, \{2,5\}, \{3,4\}, \{3,6\}, \{4,5\}\}.$$

Decide, with justification, whether or not G is bipartite. **[4]**

End of Paper.

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