Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6109 Combinatorics

Duration: 2 hours

Date and time: 29 April 2014, 1000h

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You should attempt all questions. Marks awarded are shown next to the questions.

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Examiner(s): R.W. Whitty, M. Jerrum

Question 1 (a) Four athletes A, B, C and D run in a race. They have equal abilities so that all place orderings have equal probabilities. It is assumed that no two athletes will run the race in exactly the same time (there are no ties).

(i) What is the probability that the athletes finish in alphabetical order: $ABCD$? [2]

(ii) What is the probability that A is **not** placed first? [2]

(iii) What is the probability that the first two places are taken, in either order, by A and B? [2]

(iv) The race is run three times, with independent outcomes. What is the probability that the first two places are taken, in either order, by the same two players in all three races? [5]

(b) In how many ways may we choose three letters from $\{a, b, c, d, e\}$

(i) without replacement? [1]

(ii) with replacement? [3]

(iii) with replacement except for the letter e? [5]

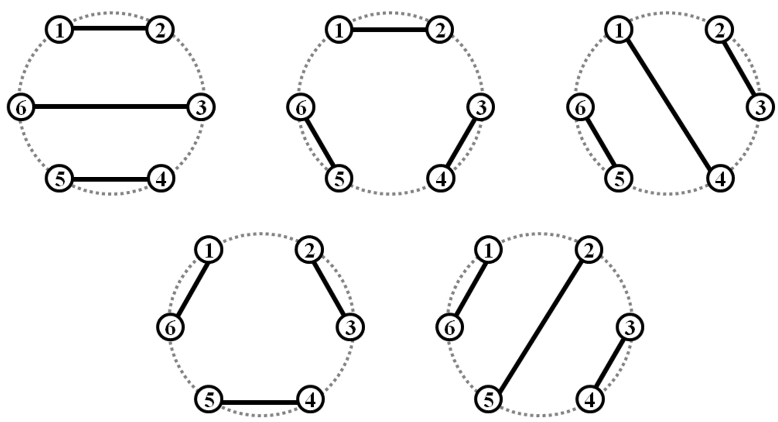
Question 2 (a) A sequence $(a_n)_{n \geq 0}$ is defined by the linear recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + 4n - 1, n \geq 2, \text{ with } a_0 = -1, a_1 = 2.$$

(i) Derive a closed form expression for a_n, for $n \geq 0$, in terms of n. [8]

(ii) Specify suitable constants c_1 and c_2 such that the growth of a_n, as $n \to \infty$, may be described in the form $c_1 n c_2^n$. [4]

(b) The **Catalan numbers** $C_n, n \geq 0$, form the sequence $1, 1, 2, 5, 14, 42, \ldots$, whose n-th entry may be defined to be the number of ways in which $2n$ points placed around a circle may be paired up by n non-intersecting straight lines. This is shown below for $n = 3$:



By using this definition and considering the possible lines drawn from point 1 on the circle, give a combinatorial proof of the recurrence [8]

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, n \geq 1.$$

Question 3 A sequence $(a_n)_{n\geq 0}$ is defined by the linear recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}, n \geq 2, \text{ with } a_0 = -1, a_1 = 3.$$

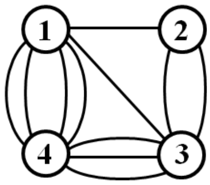
(a) Prove by strong induction on n that a_n is a multiple of 3 for all $n \geq 1$. [4]

(b) Suppose that $A(x)$ is the generating function for the sequence of a_n.

(i) Use the method of generating functions to derived a closed-form expression for $A(x)$ as a function of x. [10]

(ii) By applying a partial fractions separation to your solution to part (i), or otherwise, derive a closed-form expression for a_n, in terms of n. [6]

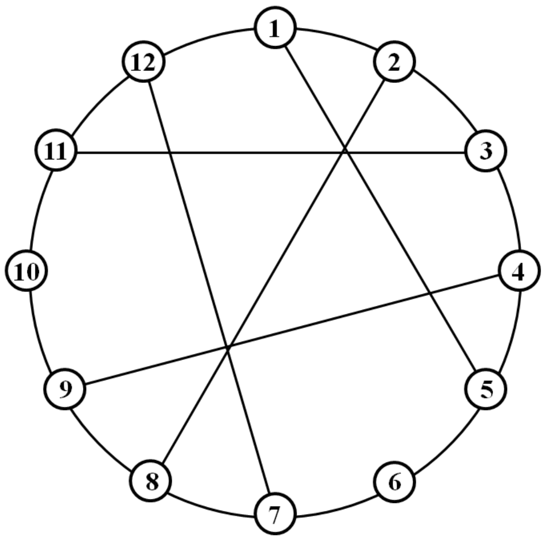
Question 4 (a) Show that the number of labelled spanning trees for the 4-vertex graph shown below is 71. [6]



(b) (i) Use Euler's Polyhedral Formula $n-e+f=2$ to show that a simple planar connected graph with n vertices and e edges and in which every cycle has length at least 5, satisfies [5]

$$e \leq \frac{5}{3}(n-2). \tag{1}$$

(ii) By using inequality (1), prove that the graph shown below is nonplanar. [4]



(iii) State Kuratowski's theorem on planarity and use this theorem to give another proof of the nonplanarity of the graph in part (ii). [5]

Question 5 (a) A set system $\mathscr{A} = \{A_i\}$ with ground set $E = \{a, b, c, d, e\}$ is specified as

$$A_1 = \{a, c\},\ A_2 = \{b, c\},\ A_3 = \{a, d, e\},\ A_4 = \{a, b, c\},\ A_5 = \{b, c\}.$$

(i) Use Hall's Theorem to show that no complete transversal (system of distinct representatives) exists for this set system. [4]

(ii) Suppose that A_1 is replaced by $A_1 = \{a, c, d\}$ to give a new set system $\mathscr{A}'$. Write down the 5×5 incidence matrix M of this new set system. [2]

(iii) Referring to $\mathscr{A}'$ and M in part (ii), specify a set of entries of M which correspond to a complete transversal of $\mathscr{A}'$. [2]

(iv) Give the value of the permanent of M. [2]

(b) (i) Write down the multiplication table for the set $\{1, 3, 5, 7\}$ working modulo 8. [3]

(ii) Explain what it means to say that your answer to part (i) forms a **standard form Latin square** on the alphabet $\{1, 3, 5, 7\}$. [2]

(iii) By identifying suitable transversals of the Latin square you identified in part (ii) specify a Latin square orthogonal to it. [5]

End of Paper