Queen Mary
University of London

Main Examination period 2017

MTH6107 / MTH6107P: Chaos & Fractals

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: O. Jenkinson

Question 1. [22 marks]

(a) How is Sharkovsky's ordering of $\mathbb{N}$ defined? **[3]**

(b) State Sharkovsky's Theorem. **[3]**

(c) Let the map $f : \mathbb{R} \to \mathbb{R}$ be given by the formula $f(x) = 1 - ax^2$, where the constant $a \approx 1.75488$ is defined to be the real solution to the equation $a(1-a)^2 = 1$.

Show that the orbit under f of the point 1 is periodic. Determine the minimal period of this orbit, and whether this orbit is unstable, stable, or superstable. **[5]**

(d) Show that f has an orbit of minimal period n for every $n \in \mathbb{N}$. **[3]**

(e) Find all of the fixed points of f, and determine whether each fixed point is unstable, stable, or superstable. **[4]**

(f) Let F denote the restriction of f to the interval $[-1,1]$ (i.e. $F : [-1,1] \to [-1,1]$ is defined by $F(x) = 1 - ax^2$).

(i) Is every periodic orbit for f also a periodic orbit for F? Justify your answer. **[2]**

(ii) Does F have an orbit of minimal period n for every $n \in \mathbb{N}$? Justify your answer. **[2]**

Question 2. [30 marks] Let $\mathscr{H}$ denote the collection of compact subsets of $\mathbb{R}$. Let $\Phi : \mathscr{H} \to \mathscr{H}$ be the iterated function system defined by the two maps $\phi_1(x) = x/5$ and $\phi_2(x) = (x+4)/5$, and let C_k denote $\Phi^k([0,1])$ for $k \geqslant 0$.

(a) For $A, B \in \mathscr{H}$, what is the definition of the **Hausdorff distance** $h(A,B)$? **[5]**

(b) Write down the sets C_1 and C_2. **[4]**

(c) Compute $h(C_1, C_2)$. **[5]**

(d) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k. **[3]**

(e) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k? **[3]**

(f) Given a set $A \subset \mathbb{R}$, what is the definition of its **box dimension**? **[5]**

(g) Using your answers to parts (d) and (e), or otherwise, show that if the box dimension of $C = \cap_{k=0}^{\infty} C_k$ exists then it must equal $\log 2 / \log 5$. **[5]**

Question 3. [22 marks]

(a) If X and Y are intervals in $\mathbb{R}$, explain what it means for two maps $f : X \to X$ and $g : Y \to Y$ to be **topologically conjugate**. [**3**]

(b) Show that if h is a topological conjugacy between f and g, then h is also a topological conjugacy between f^n and g^n, for all integers $n \geqslant 1$. [**5**]

(c) Using (b) above, or otherwise, show that if f and g are topologically conjugate then for each $n \in \mathbb{N}$, every period-n orbit for f is mapped by the conjugacy to a period-n orbit for g. [**5**]

(d) If $f : [0,\infty) \to [0,\infty)$ is defined by $f(x) = 4x$, and $g : [0,\infty) \to [0,\infty)$ is defined by $g(x) = 2x$, use the map $h(x) = \sqrt{x}$ to show that f and g are topologically conjugate. [**5**]

(e) Determine whether the map $F : [0,1) \to [0,1)$ given by $F(x) = 4x \pmod 1$ is topologically conjugate to the map $G : [0,1) \to [0,1)$ given by $G(x) = 2x \pmod 1$, being careful to justify your answer. [**4**]

Question 4. [26 marks] Suppose $f : [0,1] \to [0,1]$.

(a) If f is C^1, what is the definition of the **Lyapunov exponent** $\lambda(x)$ of f at $x \in [0,1]$? [**3**]

(b) If f is C^1 and x is a point of minimal period N, what is the definition of its **multiplier**? [**2**]

(c) Use the Intermediate Value Theorem to show that if f is continuous then it has at least one fixed point. [**8**]

(d) Show that if f is continuous and order reversing (i.e. $f(x) > f(y)$ whenever $x, y \in [0,1]$ satisfy $x < y$) then f has a unique fixed point (you may use the result from part (c) above). [**5**]

(e) Does there exist a continuous map $g : (0,1) \to (0,1)$ which has no fixed points? Justify your answer. [**4**]

(f) Does there exist a discontinuous order reversing map $h : [0,1] \to [0,1]$ which has no fixed points? Justify your answer. [**4**]

End of Paper.