Queen Mary University of London

Main Examination period 2021 – January – Semester A

MTH6106 / MTH6106P: Group Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: Matthew Fayers and Alex Fink

In this paper, we use the following notation.

- $\mathcal{U}_n$ is the set of integers between 0 and n which are prime to n, with the group operation being multiplication modulo n.
- $\mathcal{D}_{2n}$ is the group with $2n$ elements
$$1,\ r,\ r^2,\ \ldots,\ r^{n-1},\ s,\ rs,\ r^2s,\ \ldots,\ r^{n-1}s.$$
The group operation is determined by the relations $r^n = s^2 = 1$ and $sr = r^{n-1}s$.
- $\mathcal{S}_n$ denotes the group of all permutations of $\{1, \ldots, n\}$ (with the group operation being composition) and $\mathcal{A}_n$ is the subgroup of $\mathcal{S}_n$ consisting of all even permutations.
- $\mathrm{GL}_2(\mathbb{R})$ is the group of invertible 2×2 matrices over $\mathbb{R}$, with the group operation being matrix multiplication.

Question 1 [16 marks].

(a) Suppose G is a set with three elements a, b, c, with a binary operation given by the following table.

	a	b	c
a	b	a	c
b	a	b	c
c	c	c	b

Which of the group axioms G1–G4 does G satisfy? Justify your answer. **[5]**

(b) Now let
$$H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R},\ a^2 + b^2 \neq 0 \right\} \cup \left\{ \begin{pmatrix} c & d \\ d & -c \end{pmatrix} \,\middle|\, c, d \in \mathbb{R},\ c^2 + d^2 \neq 0 \right\}.$$
Prove that H is a subgroup of $\mathrm{GL}_2(\mathbb{R})$. **[6]**

Now suppose G is a group. Recall that if $g \in G$, the **order** of g is the smallest positive integer n such that $g^n = 1$, or ∞ if no such n exists.

(c) Suppose $f, g \in G$ satisfy $gf = f^{-1}g$ and $\mathrm{ord}(g) = 4$. What is $\mathrm{ord}(fg)$? Justify your answer. **[5]**

Question 2 [23 marks]. Suppose G is a group.

(a) Suppose $f, g \in G$ and $H \leqslant G$, and that $f \in gH$. Prove that $fH = gH$. **[5]**

(b) Let $G = \mathcal{U}_{36}$, and let $H = \{1, 17, 19, 35\}$. List the elements of each right coset of H in G. [You may assume $H \leqslant G$.] **[5]**

(c) Let x be the 4th digit of your student number. Suppose $H \leqslant \mathcal{D}_{12}$ and $g \in \mathcal{D}_{12}$ such that

$$Hg = \{r, r^x s, r^4, r^{3+x} s\}.$$

Find all the elements of H. [Explain your reasoning.] **[5]**

For the rest of this question, let G be the following group of order 21:

$$G = \left\{1, a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, b^2, ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2\right\},$$

where a is an element of order 7, b is an element of order 3 and $ba = a^4b$.
Let y be the last digit of your student number that isn't 0 or 7. (For example, if your student number is 180184370, then $y = 3$.) Let $g = a^y b$.

(d) Find the elements of $\langle g \rangle$, writing them in the form $a^i b^j$ as in the above list. [You may assume that $ba^r = a^{4r} b$ for every $r \in \mathbb{Z}$.] **[5]**

(e) Is $\langle g \rangle$ a normal subgroup of G? Justify your answer. **[3]**

Question 3 [14 marks].

(a) Let $f, g \in \mathcal{S}_6$ be the permutations written in two-line notation as follows.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 6 & 2 & 5 & 1 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 2 & 6 & 3 & 4 & 1 \end{pmatrix}.$$

Write f, g^{-1} and fgf^{-1} in disjoint cycle notation. **[5]**

(b) Write the permutation $h = (1\,3)(2\,4\,6)(5\,7\,9)$ as a product of transpositions, and hence decide whether h is even or odd. **[4]**

(c) Does $\mathcal{S}_8$ contain an element of order 12? Does $\mathcal{A}_8$ contain an element of order 12? Justify your answers, saying which results from the lectures you are using. **[5]**

Question 4 [13 marks]. Suppose G is a group. Recall that an **automorphism** of G is a bijection $\phi : G \to G$ such that $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in G$.
Recall that for $g \in G$ we define the automorphism

$$\begin{aligned} \rho_g : G &\longrightarrow G \\ h &\longmapsto ghg^{-1}. \end{aligned}$$

Finally recall that $Z(G)$ denotes the centre of G.

(a) Given $f, g \in G$, show that $\rho_f = \rho_g$ if and only if $f^{-1}g \in Z(G)$. **[5]**

(b) Suppose $G = \mathcal{D}_8$. Find $g \in \mathcal{D}_8$ such that

$$\rho_g(r) = r^3, \qquad \rho_g(s) = r^2 s.$$ **[4]**

(c) Find an automorphism ϕ of $\mathcal{D}_8$ such that $\phi(s) = r^3 s$. [You should say where each element of $\mathcal{D}_8$ maps to, but you do not need to prove anything.] **[4]**

Question 5 [22 marks].

(a) A student tried to write down the definition of an action as follows.

> Suppose G is a group and X is a set. An **action** of G on X is a function $\pi : G \to X$ satisfying the following axioms.
>
> (A1) $\pi_1 = \mathrm{id}_X$;
>
> (A2) $\pi_f \circ \pi_g = \pi_g \circ \pi_f$ for all $f, g \in X$.

The student's definition is wrong in three ways. What are the three problems with the definition? **[3]**

(b) Give an example of an action of $\mathcal{S}_3$ on $\mathcal{S}_3$ which has exactly three orbits. Explain briefly why there are three orbits. **[4]**

(c) Suppose G is a group and $H \leqslant G$. Let X be the set of right cosets of H in G, and define

$$\pi_g(Hk) = H(kg^{-1}) \qquad \text{for all } g, k \in G.$$

Prove that π_g is well-defined for each g, and that this defines an action. **[8]**

(d) Suppose we colour the vertices and edges of a square, and we have n colours available. We say that two colourings are equivalent if we can transform one into the other by applying a symmetry of the square. How many colourings are there up to equivalence? [You should carefully explain the method you use, and which results you use from lectures, as well as carrying out the calculation.] **[7]**

Question 6 [12 marks].

(a) Write down two groups of order 27 which are not isomorphic. Explain briefly how you know they are not isomorphic. [You should define your groups carefully, saying what the underlying set and group operation are in each case, but you do not have to prove that they are groups.] **[6]**

Now suppose G is a finite group and p is a prime number. Recall that G' is the commutator subgroup of G.

(b) Suppose G is a finite p-group and $G \neq \{1\}$. Using results from the lecture notes, show that $G' \neq G$. [Hint: consider a composition series for G.] **[6]**

End of Paper.