M. Sc. Examination by course unit 2014

MTH6104P: Algebraic structures II

Duration: 2 hours

Date and time: 16th May 2014, 10:00 a.m.

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best four questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

Important note: the Academic Regulations state that possession of unauthorised material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QMUL.

Please check now to ensure you do not have any notes, mobile phones or unauthorised electronic devices on your person. If you have any, then please raise your hand and give them to an invigilator immediately. Please be aware that if you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. Disruption caused by mobile phones is also an examination offence.

Exam papers must not be removed from the examination room.

Examiner(s): Matthew Fayers

In this paper, we use the following notation.

- C_n denotes the cyclic group of order n.
- Q_8 is the group $\{1, -1, i, -i, j, -j, k, -k\}$, in which
$$i^2 = j^2 = k^2 = -1, \qquad ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$
- $\mathrm{GL}_2(\mathbb{C})$ is the group of invertible 2×2 matrices over $\mathbb{C}$, with the group operation being matrix multiplication.

Question 1

(a) Give the definition of a *group*, and a *subgroup*. [5]

Suppose G is a group and $g \in G$.

(b) Give the definition of the *cyclic subgroup of G generated by g*, and prove that it is a subgroup of G. *[You may use elementary rules for manipulating powers of elements.]* [6]

(c) Give the definition of the *order* of g. [2]

For the rest of this question, let G be the following group of order 21:

$$G = \left\{1, a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, b^2, ab^2, a^2b^2, a^3b^2, a^4b^2, a^5b^2, a^6b^2\right\},$$

where a is an element of order 7, b is an element of order 3 and $ba = a^2b$.

(d) Find three elements of G which are conjugate to ab. [5]

(e) Find the elements of $\langle ab \rangle$. [4]

[For parts (d) and (e) you should write elements of G in the form a^ib^j as in the above list.]

(f) Is $\langle ab \rangle$ a normal subgroup of G? Justify your answer. [3]

Question 2

(a) Give the definition of the *symmetric group* $\mathcal{S}_n$. [3]

(b) Suppose $f, g \in \mathcal{S}_5$ are defined by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}.$$

Write down g, g^{-1} and fgf^{-1} in disjoint cycle notation. [4]

(c) Give the definition of a *transposition*. [2]

(d) Give the definition of the *alternating group* $\mathcal{A}_n$, and prove that $\mathcal{A}_n \trianglelefteq \mathcal{S}_n$. [7]

(e) Suppose $h \in \mathcal{S}_n$. Explain how you can use the disjoint cycle notation for h to find the order of h and to find whether $h \in \mathcal{A}_n$. [3]

(f) Suppose $N \trianglelefteq \mathcal{A}_6$, and that N contains the element $(1\ 3\ 5)(2\ 4\ 6)$. Prove that N contains a 3-cycle. *[You may not assume that $\mathcal{A}_6$ is simple.]* [6]

Question 3 Suppose G is a group.

(a) Suppose X is a set. Give the definition of an *action* of G on X. [3]

(b) Suppose n is a positive integer, and let X be the set of all subsets of G of size n. Define an action of G on X by

$$\pi_g(\{g_1, \dots, g_n\}) = \{gg_1, \dots, gg_n\}.$$

Prove that π really is an action. [5]

(c) Suppose π is an action of G on X, and $x \in X$. Give the definition of the *orbit* of x and the *stabiliser* of x. [4]

(d) State and prove the Orbit–Stabiliser Theorem. *[You may assume that a stabiliser is a subgroup, and you may use Lagrange's Theorem.]* [8]

(e) Use the Orbit–Stabiliser Theorem to find the order of the symmetry group of a cube. [5]

Question 4 *[In this question, you may assume any results you need about actions of groups.]*
Suppose p is a prime.

(a) What does it mean to say that a finite group is a *p-group*? [2]

(b) Describe a group G of order p^2 which is not isomorphic to C_{p^2}. *[You should be explicit about what the underlying set and the binary operation are, but you do not have to prove that G is a group.]* Explain how you know that G is not isomorphic to C_{p^2}. [4]

(c) Give the definition of a *Sylow p-subgroup*. [2]

(d) Give precise statements of all the Sylow Theorems. [7]

(e) Using one of the Sylow Theorems, show that:

- there is only one group of order 87 up to isomorphism, and
- there is no simple group of order 56.

[You may assume a general result relating the order of a group to the orders of its elements.] [10]

Question 5 Suppose G is a group, $g \in G$ and $N, H \leqslant G$.

(a) Given the definition of the *left coset* gH. [2]

(b) Prove that if $a, b, c \in gH$ then $ab^{-1}c \in gH$. [3]

(c) Give the definition of the set NH. [2]

(d) Prove that if $N, H \trianglelefteq G$, then $NH \trianglelefteq G$. [7]

(e) Give an example of a group G with $N, H \leqslant G$ such that NH is not a subgroup of G. *[You do not have to prove that N and H are subgroups, but you should prove that NH is not a subgroup.]* [5]

(f) Now suppose $G = \mathrm{GL}_2(\mathbb{C})$, and let

$$N = \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle, \qquad H = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle.$$

Find all the left cosets of H in NH (which is a subgroup of G in this case). [6]

Question 6 Suppose G and H are groups.

(a) Give the definitions of the following:

- a *homomorphism* from G to H;
- an *isomorphism* from G to H;
- an *automorphism* of G;
- the *automorphism group* of G. [5]

(b) Suppose $\phi : G \to H$ is a homomorphism, and $L \leqslant H$. Prove that $\phi(\phi^{-1}(L)) = L \cap \operatorname{Im}\phi$. [4]

(c) Give a precise statement of the Correspondence Theorem. [4]

(d) Let $G = C_{50}$. Find all the subgroups of G, and draw a diagram showing which subgroups contain which others. *[You do not have to prove anything.]* [4]

(e) Let $\phi : C_{50} \to C_{50}$ be the homomorphism which sends g to g^5 for every $g \in C_{50}$. Find $\operatorname{Im}\phi$ and $\ker(\phi)$, and show how subgroups correspond under the Correspondence Theorem. *[You do not have to prove anything.]* [4]

(f) Give an example of an outer automorphism ϕ of Q_8 such that $\phi(i) = i$. *[You do not have to prove anything, but should you say where each element of Q_8 maps to.]* [4]

End of Paper