Queen Mary University of London

Main Examination period 2017

MTH5121: Probability Models

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: I. Goldsheid

Question 1. [34 marks] Suppose that X and Y are two random variables.

(a) Define what it means for two random variables X and Y to be *independent*. **[3]**

(b) Suppose that the joint density function $f_{X,Y}$ is given by

$$f_{X,Y}(x,y) = \begin{cases} cxy & \text{if } 0 < x < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Prove that $c = \frac{1}{2}$. **[5]**

(ii) Show that the marginal probability density function for Y is

$$f_Y(y) = \begin{cases} \frac{1}{4}y^3 & \text{if } y \in (0,2), \\ 0 & \text{otherwise.} \end{cases}$$ **[8]**

(iii) Find the probability $\mathbb{P}\{X \leqslant 1 \text{ and } Y \leqslant 1\}$. **[7]**

(iv) Are X and Y independent? Justify your answer. **[3]**

(v) Find the conditional density function $f_{X|Y=y}(x)$ and compute the conditional expectation $\mathbb{E}(X^2|Y)$. **[8]**

Question 2. [16 marks] Suppose that a random walk on a line starts from n, $M \leqslant n \leqslant N$. The probability of a jump to the right is p and the probability of a jump to the left is $q = 1 - p$. The walk stops once it reaches M or N. Let r_n be the probability that the walk starting from n reaches N before M and E_n be the expected duration of the walk.

(a) Write down the equations for r_n, where $M \leqslant n \leqslant N$. **[3]**

(b) Write down the equations for E_n, where $M \leqslant n \leqslant N$. **[3]**

(c) Suppose now that $p = q = 1/2$ and $M = 0$. Write down the solution to the equations from part (b) (no proof is required). **[2]**

(d) Now take $N = \infty$, so that the random walk is on the whole of the non-negative integers. Prove the following statement: Suppose that $p = q = 1/2$ and the random walk starts from position 2. Then the expected time until it reaches zero is infinite. **[8]**

Question 3. [21 marks] Let $Y_0, Y_1, Y_2 \ldots$ be a branching process starting with one ancestor, (that is $Y_0 = 1$) and generated by a random variable X with the mean value $\mathbb{E}(X)$.

(a) State, in terms of the mean value of X, the necessary and sufficient condition for the probability of extinction of the branching process to be strictly less than 1. **[2]**

(b) Suppose now that X has distribution $\mathbb{P}(X = k) = pq^k$, $k = 0, 1, 2, ...$, where $0 < p < 1$ and $q = 1 - p$.

(i) Compute the probability generating function $G(t)$ of the random variable X and show that $\mathbb{E}(X) = \frac{q}{p}$. **[4]**

(ii) Prove that the probability of extinction of this branching process is 1 if and only if $p \geqslant 0.5$.
Hint: use the statement of question 3(a). **[4]**

(iii) Suppose now that $p < 0.5$. Find the probability of extinction of this branching process. **[7]**

(iv) Prove that

$$\mathbb{P}(Y_{10} \geqslant 100) \leqslant \frac{q^{10}}{100p^{10}}.$$

Hint: use Markov's inequality. **[4]**

Question 4. [11 marks] Let $N(t)$ be a Poisson process.

(a) Give the definition of the Poisson process $N(t)$ with rate $\lambda > 0$. **[5]**

(b) Prove that if $0 < t < s$ and $m,\ n$ are integers such that $0 \leqslant m \leqslant n$ then

$$\mathbb{P}(N(t) = m,\ N(s) = n) = e^{-\lambda s}\lambda^n \frac{t^m(s-t)^{n-m}}{m!(n-m)!}$$ **[6]**

Question 5. [18 marks]

(a) State and prove Markov's inequality. **[8]**

(b) State (do not prove) the Law of Large Numbers. **[5]**

(c) State the Central Limit Theorem. **[5]**

End of Paper.