B. Sc. Examination by course unit 2014

MTH5121 Probability Models

Duration: 2 hours

Date and time: 10.00am, 6th May 2014

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): M. J. Walters

Unless explicitly stated otherwise you should justify all your answers. You may use any theorems from the course but you should make it clear which result you are using.

Question 1 (12 Marks)
Suppose that X is a random variable with probability generating function

$$G_X(t) = \frac{t}{2} + \frac{1}{2}e^{t-1}.$$

(a) Find $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$ and $\mathbb{P}(X = 2)$. [6]

(b) Find the mean and variance of X. [6]

Question 2 (13 Marks)

(a) State the Theorem of Total Expectation. [5]

(b) I play a game. On my turn I roll a fair dice, and I move my counter that many spaces. If the dice comes up 6 my turn *ends* after I have moved my counter. Otherwise I roll again following the same rules. Find the expected number of spaces I move on my turn. [8]

Question 3 (15 Marks)

(a) Suppose that X is a random variable. State the definition that a sequence of random variables $Y_0, Y_1, Y_2, Y_3 \ldots$ form the branching process generated by X. [5]

(b) **For this part of the question you may use any theorem from the course but you MUST write out its full statement.**

Suppose that the probability generating function for X is $G_X(t) = \frac{1}{5} + \frac{2}{5}t + \frac{2}{5}t^2$. Find the probability of extinction. [10]

Question 4 (10 Marks)

(a) State Markov's Inequality. [4]

Suppose that $X_1, X_2, X_3, \ldots$ are independent Bernoulli random variables with parameter p and that N is an independent Poisson random variable with mean λ. Let $X = \sum_{i=1}^{N} X_i$

(b) State the mean of X. [2]

(c) Show that $\mathbb{P}(X \geqslant \lambda) \leqslant p$. [4]

Question 5 (15 Marks)
Suppose that X and Y are jointly distributed random variables with joint density function $f_{X,Y}$ given by

$$f_{X,Y}(x,y) = \begin{cases} Ce^{-x-3y} & \text{if } x > y > 0 \\ 0 & \text{otherwise} \end{cases}$$

for some constant C.

(a) Show that $C = 4$. [7]

(b) Find the probability that $2 \leqslant X \leqslant 3$ and $0 \leqslant Y \leqslant 1$. You may leave your answer as integral provided that you have simplified it so that the integral does not involve $f_{X,Y}$. [4]

(c) Find the probability that $0 \leqslant X \leqslant 1$ and $2 \leqslant Y \leqslant 3$. [4]

Question 6 (15 Marks)

(a) State the approximate central limit theorem. [5]

I travel to work and back each weekday for 10 weeks (so 100 trips in total). Each trip has mean length 30 minutes and variance 36 minutes, and the lengths of the trips are independent.

Let S be the total time I spend travelling to and from work over these 10 weeks.

(b) What is the approximate distribution of S? [5]

(c) Find in terms of the cumulative normal distribution Φ the (approximate) probability that S is between 49 hours (2940 minutes) and 52 hours (3120 minutes). [5]

Question 7 (20 Marks)
Suppose that $T_1, T_2, T_3 \ldots$ are the arrival times of a Poisson Process of rate λ.

(a) Derive the density function $f_{T_1}(t_1)$ for the first arrival time T_1 directly from the definition of the Poisson Process. [6]

We showed in lectures that the joint density function of T_1 and T_2 is

$$f_{T_1,T_2}(t_1,t_2) = \begin{cases} \lambda^2 e^{-\lambda t_2} & \text{if } t_2 \geqslant t_1 \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Find the conditional density $f_{T_2|T_1=t_1}$. [6]

(c) Find $\mathbb{E}(T_2 \mid T_1 = t_1)$. [8]

End of Paper