Exam

School of Mathematical Sciences

MTH5110 Introduction to Numerical Computing

6 May 2016, 10:00–12:00 **Duration: 120 minutes**

Answer ALL questions

Answer each question in the appropriate subsection headed *Answers* in the Maple Worksheet provided in QMplus. Depending on the problem, you must use Maple to perform the calculations or use Maple to write down your computations made by hand. Do not delete any relevant input or output; you will score marks only for what is visible in the document you submit. There may be more than one correct solution to each question; any working solution will be accepted provided it satisfies the requirements of the question.

This exam is open book. You may access any information you want, but must work entirely by yourself. You may not communicate, nor attempt to communicate, with anyone else, nor solicit assistance in any way. Please be aware that details of all internet activity on your computer may be logged. You may do rough work on your own paper, which will not be collected by the invigilators.

Examiners: J. Starke, F. J. Wright

Problem 1 [total 25 marks for Problem 1]

This problem is about investigating the map from x to $h(x)$ for $x \in \mathbb{R}$ that represents the simplified Newton method and comparing it with the map from x to $g(x)$ of the original Newton method to find a root x^* of a function $f: \mathbb{R} \to \mathbb{R}$ and $f \in C^2$. The two methods should use the same seed x_0.

(You may use Maple to perform the computation or you may perform it by hand and use Maple to write it down.)

a) [5 marks]
Write down the map from x to $g(x)$ required for the original Newton method.

b) [5 marks]
Write down the map from x to $h(x)$ required for the simplified Newton method.

c) [5 marks]
Compute the derivative $\frac{d}{dx} g(x)$ and evaluate it at the root x^*.

d) [5 marks]
Compute the derivative $\frac{d}{dx} h(x)$ and evaluate it at the root x^*.

e) [5 marks]
Explain briefly (in 1–3 sentences), by using the results from c) and d), why the original Newton method with $g(x)$ is a second order method and why the simplified Newton method with $h(x)$ converges slower than the original Newton method.

Answers

Before starting this question, execute this command:

```
> restart;
>
```

Now save this document.

Problem 2 [total 25 marks for Problem 2]

The following Maple procedure computes a numerical solution to a given ordinary differential equation $\frac{d}{dt}x = f(x)$ with $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$
by using an Euler forward method. The procedure is tested with $f(x) = x$, initial value $x(0) = 1$ and time period $[0, 5]$.

a) [6 marks]
Give the formula of the Euler forward method to compute a point x(t+Δ t) from a given point x(t).

b) [6 marks]
Give the formula of the modified Euler method (midpoint method) to compute a point x(t+2· Δ t) from a given point x(t).

c) [6 marks]
Change the ODE solver in the given Maple procedure from the Euler forward method to the modified Euler method.

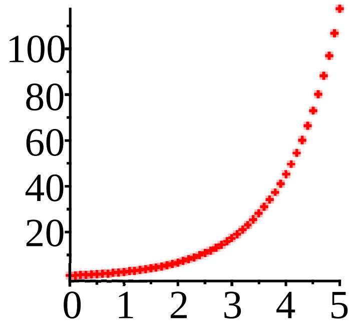
d) [7 marks]
Compare the results of the two ODE solvers (Euler forward and modified Euler) by plotting the results in one diagram for the step sizes 0.1 and 0.01 between the discrete times of computed points. Give arguments why one of the two methods outperforms the other one.

```
> Eulerforward := proc (t0::numeric, t1::numeric, x0::numeric,
  f::procedure, Deltat::numeric)
  # t0: initial time
  # t1: end time
  # x0: initial value
  # f: rhs of ODE
  # Deltat: step size for Euler method
    local xold, xnew, tnew, soln;
    xnew := x0;
    tnew := t0;
    soln := [t0, x0];
    while tnew < t1 do
      tnew := tnew+Deltat;
      xold := xnew;
      xnew := xold+Deltat*f(xold);
      soln := soln, [tnew, xnew];
    end do;
    return [soln];
  end proc:
```

```
> listresult := Eulerforward(0, 5.0, 1.0, x->x, 0.1);
```

listresult := [[0, 1.0], [0.1, 1.10], [0.2, 1.210], [0.3, 1.3310], [0.4, 1.46410], [0.5, 1.610510], [0.6, 1.7715610], [0.7, 1.94871710], [0.8, 2.143588810], [0.9, 2.357947691], [1.0, 2.593742460], [1.1, 2.853116706], [1.2, 3.138428377], [1.3, 3.452271215], [1.4, 3.797498336], [1.5, 4.177248170], [1.6, 4.594972987], [1.7, 5.054470286], [1.8, 5.559917315], [1.9, 6.115909046], [2.0, 6.727499951], [2.1, 7.400249946], [2.2, 8.140274941], [2.3, 8.954302435], [2.4, 9.849732678], [2.5, 10.83470595], [2.6, 11.91817654], [2.7, 13.10999419], [2.8, 14.42099361], [2.9, 15.86309297], [3.0, 17.44940227], [3.1, 19.19434250], [3.2, 21.11377675], [3.3, 23.22515442], [3.4, 25.54766986], [3.5, 28.10243685], [3.6, 30.91268054], [3.7, 34.00394859], [3.8, 37.40434345], [3.9, 41.14477780], [4.0, 45.25925558], [4.1, 49.78518114], [4.2, 54.76369925], [4.3, 60.24006918], [4.4, 66.26407610], [4.5, 72.89048371], [4.6, 80.17953208], [4.7, 88.19748529], [4.8, 97.01723382], [4.9, 106.7189572], [5.0, 117.3908529]] **(2.1)**

```
> plotresult := plot(Eulerforward(0, 5.0, 1.0, x->x, 0.1), style = point, colour = red);
```

plotresult := 

Answers

Before starting this question, execute this command:

```
> restart;
>
```

Now save this document.

Problem 3 [total 25 marks for Problem 3]

The purpose of this question is to investigate the fixed point iteration $x_{n+1} = f(x_n)$ with f(x)= $\frac{4}{5}$ x - 3.

a) [7 marks]
Compute the fixed point of this map analytically (not numerically) with Maple or by hand.

b) [9 marks]
Compute $x_1, x_2, \ldots, x_{30}$ of the fixed point iteration numerically with seed $x_0 = -14$ and verify with that your result from a).

c) [9 marks]
Explain why this fixed point iteration converges to the fixed point by investigating k-contractivity with the distance d(x,y)=|x-y| .

Answers

Before starting this question, execute this command:

```
> restart;
>
```

Now save this document.

Problem 4 [total 25 marks for Problem 4]

The circumference of an ellipse with semi-major axis a and semi-minor axis b can be computed with the definite integral

$$C = 4\left(\int_0^{\frac{\pi}{2}} \sqrt{(a\sin(t))^2 + (b\cos(t))^2}\, \mathrm{d}t\right).$$

a) [8 marks]
Write a Maple procedure which computes a numerical approximation of this integral using the trapezoidal rule. Your procedure should have as input the semi-major axis a, the semi-minor axis b, and the number of subintervals n. Your procedure should return the numerical approximation of C. Use your procedure to compute C for an ellipse with $a = 3$ and $b = 2$ using 100 subintervals.

b) [8 marks]
Add to your program the computation of the numerical approximation of C also as Riemann sum. Write a new Maple procedure which produces a plot where you show the absolute value of the difference between the results of the two numerical integration methods over n for $n = 2, 3, 4, \ldots, 50$. Which method is the better choice and why?

c) [9 marks]
Use your result from b) to determine how the difference between the two methods depend on n. You should give an approximate formula for this dependence. You can determine the parameters from a plot. Hint: A loglogplot is useful.

Answers

Before starting this question, execute this command:

```
> restart;
>
```

Now save this document.

Now save this document.

End of Paper