Queen Mary
University of London

Main Examination period 2017

MTH5109: Geometry II: Knots and Surfaces

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

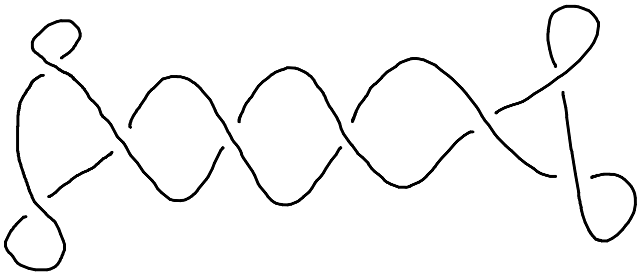
Exam papers must not be removed from the examination room.

Examiners: A. Shao

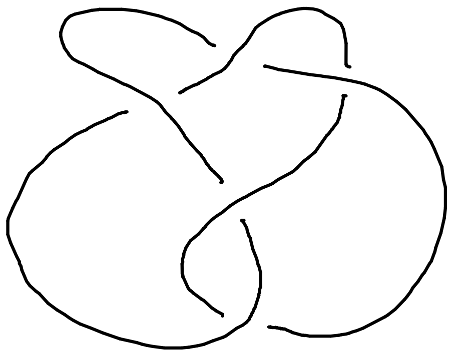
Question 1. [**17 marks**]

(a) List three different knot invariants. [**3**]

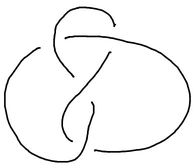
(b) Is the knot below tricolourable? Justify your answer. [**4**]

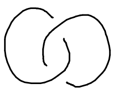


(c) Compute the Kauffman bracket of the following knot diagram:



You may use without proof the following identities,

$B($  $, x) = x^8 - x^4 + 1 - x^{-4} + x^{-8},$

$B($  $, x) = -x^4 - x^{-4},$

where B denotes the Kauffman bracket. (It may be simplest to work first with the crossing at the top centre of the given diagram.) [**6**]

(d) Suppose a knot K has Jones polynomial

$$J(K, t) = t^2 - t + 2 - 2t^{-1} + t^{-2} - t^{-3} + t^{-4}.$$

Is K chiral? Justify your answer. [**4**]

Question 2. [16 marks]

(a) For the parametrised curve

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \left(\cos t, \frac{1}{2}\sin(2t)\right),$$

compute its signed curvature at every point. [**4**]

(b) For the parametrised curve

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = \left(t, \frac{1}{2}t^2, t\right),$$

compute its torsion at each point. [**4**]

(c) Find an arc length reparametrisation of the parametrised curve

$$\gamma : (0, \pi) \to \mathbb{R}^2, \qquad \gamma(t) = (2\cos(2t), 2\sin(2t)).$$

What is the domain of this reparametrisation? [**4**]

(d) Recall that the helix

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\cos t, \sin t, t)$$

satisfies the following:

$$\gamma(0) = (1, 0, 0), \qquad \gamma'(0) = (0, 1, 1), \qquad \kappa = \tau = \frac{1}{2}.$$

Is there another different curve $\lambda : \mathbb{R} \to \mathbb{R}^3$ that also satisfies

$$\lambda(0) = (1, 0, 0), \qquad \lambda'(0) = (0, 1, 1), \qquad \kappa = \tau = \frac{1}{2}?$$

Explain your answer. [**4**]

Question 3. [17 marks] Consider the surface $\mathcal{Q}$ that is defined as the image of a single parametrisation

$$\sigma : (0,3) \times \mathbb{R} \to \mathbb{R}^3, \qquad \sigma(u,v) = ((3-u)\cos v, (3-u)\sin v, u).$$

(a) Show that σ is regular. [**4**]

(b) Sketch $\mathcal{Q}$, and indicate clearly some curves of constant u as well as some curves of constant v. [**4**]

(c) Show that the first fundamental form of $\mathcal{Q}$ with respect to σ is

$$F_I^\sigma = \begin{bmatrix} 2 & 0 \\ 0 & (3-u)^2 \end{bmatrix}.$$ [**3**]

(d) Find the surface area of $\mathcal{Q}$. [**6**]

Question 4. [15 marks] Let $\mathcal{Q}$ and σ be as in Question 3.

(a) Show that the second fundamental form of $\mathcal{Q}$ with respect to σ is

$$F_{II}^\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \frac{3-u}{\sqrt{2}} \end{bmatrix}.$$ [**7**]

(b) Show that the Weingarten matrix of $\mathcal{Q}$ with respect to σ is given by

$$W^\sigma = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{2}(3-u)} \end{bmatrix}.$$ [**4**]

(c) Compute the mean and Gauss curvatures of $\mathcal{Q}$. [**4**]

Question 5. [14 marks]

(a) Suppose $S \subseteq \mathbb{R}^3$ is a surface, with γ a curve on S. Give two ways that you can check that γ is a geodesic on S. [**4**]

(b) For a surface $S \subseteq \mathbb{R}^3$ and a parametrisation σ of S, state the geodesic equations on S with respect to σ. Be sure to define any functions that you may have written down. [**4**]

(c) Let $\mathcal{T}$ be the torus parametrised by

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u),$$

and consider the parametrised curve

$$\gamma : \mathbb{R} \to \mathcal{T}, \qquad \gamma(t) = (3 \cos t, 3 \sin t, 0).$$

Show that γ is a geodesic on $\mathcal{T}$. [**6**]

Question 6. [10 marks] Consider the cylinder

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 4\},$$

which can be parametrised by

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \sigma(u, v) = (2 \cos u, 2 \sin u, v).$$

(a) Compute the **unsigned** normal curvature in $\mathcal{C}$ of the curve

$$\gamma : \mathbb{R} \to \mathcal{C}, \qquad \gamma(t) = (2 \cos t, 2 \sin t, 2 \sin t).$$ [**7**]

(b) How are the principal curvatures κ_1 and κ_2 and the principal directions $\vec{v}_1, \vec{v}_2$ of $\mathcal{C}$ related to the signed normal curvature $\kappa_{n,s}$ of γ via **Euler's theorem**? (You do not have to state Euler's theorem in its entirely, just the formula relating the above quantities.) [**3**]

Question 7. [11 marks]

(a) Consider the Möbius strip $\mathcal{M}$. What problem would you encounter if you tried to define the principal curvatures smoothly on $\mathcal{M}$? [**3**]

(b) Let X and Y be the plane and paraboloid given by

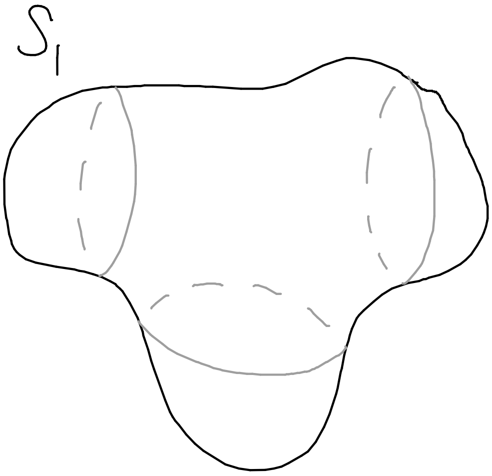
$$X = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}, \qquad Y = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}.$$

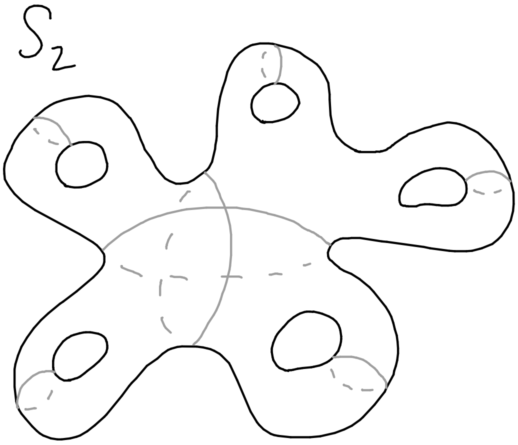
Can X be transformed into Y without "stretching" (i.e., altering distances and angles)? Explain why or why not. [**4**]

(c) For the surfaces S_1, S_2 below, use the Gauss–Bonnet theorem to find

$$\int_{S_i} \mathcal{K}_{G,i} dA, \qquad 1 \le i \le 2,$$

where $\mathcal{K}_{G,i}$ denotes the Gauss curvature of S_i. [**4**]





End of Paper.