Queen Mary
University of London

MTH5109: Geometry II: knots and surfaces

Duration: 2 hours

Date and time: 3rd May 2016, 2:30–4:30 pm

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): S. Majid

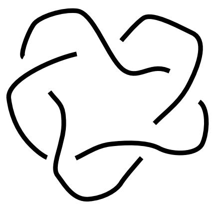
Question 1.

(a) Is the knot depicted by  chiral? Justify your answer. **[2]**

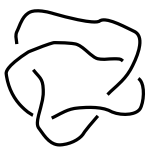
(b) Define (i) the *writhe* W of a knot diagram; (ii) the *Kauffman bracket* $B(x)$ of a knot or link diagram; (iii) the *Jones polynomial* $J(t)$ of a knot. **[9]**

(c) State how $B(x)$ behaves under a Reidemeister move of type I. **[2]**

(d) Compute $J(t)$ for the knot with diagram



You may assume (c) and that

$B($  $) = -x^{10} + x^6 - x^2 - x^{-6}.$

[8]

Question 2. Let γ be a parametrised curve in $\mathbb{R}^3$ with curvature K and torsion T.

(a) State the Serret-Frenet equations for the unit tangent vector $\mathbf{t}$, the principal normal $\mathbf{n}$ and the conormal $\mathbf{b}$ of γ. **[4]**

(b) Suppose that γ has constant $||\dot{\gamma}|| = \sqrt{2}$ and $K = T = \frac{1}{2}$. Show using (a) that $\ddot{\mathbf{n}} = -\mathbf{n}$. **[3]**

(c) You may assume without proof that the equation in (b) has general solution $\mathbf{n} = \mathbf{A}\cos t + \mathbf{B}\sin t$ for some constant orthogonal unit vectors $\mathbf{A}, \mathbf{B}$. Show using (a) that $\dot{\gamma} = \mathbf{A}\sin t - \mathbf{B}\cos t + \mathbf{C}$ for some $\mathbf{C}$ orthogonal to $\mathbf{A}, \mathbf{B}$. **[5]**

(d) Hence, or otherwise, show that γ has the shape of a circular helix. **[3]**

Question 3. Let $\gamma = (t^3, \sin t)$ be a curve in the $x-y$-plane, where $t \in (0, \frac{\pi}{2})$.

(a) Sketch the curve. **[5]**

(b) Compute $\dot{\gamma}(t)$ as $t \to 0^+$ and as $t \to \frac{\pi}{2}^-$, and mark the directions of these vectors on your sketch. State the accumulated angular change in direction on going along the curve. **[3]**

(c) Compute the signed curvature K_S of the curve. **[4]**

(d) Using parts (b)-(c) and a general result from Lectures, or otherwise, prove that

$$\int_0^{\frac{\pi}{2}} \frac{2t\cos t + t^2 \sin t}{9t^4 + \cos^2 t} \mathrm{d}t = \frac{\pi}{6}.$$

[4]

Question 4. Let S be a surface of revolution with surface patch

$$\sigma(u,v) = \frac{2}{3}\left((1-u)^{3/2}\cos v, (1-u)^{3/2}\sin v, u^{3/2}\right), \quad 0 < u < 1,\ 0 < v < 2\pi.$$

(a) Sketch the surface. **[5]**

(b) Compute the normal $\mathbf{N}$ and indicate it at a typical point on your sketch. **[5]**

(c) Compute the 1st and 2nd fundamental forms F_I, F_{II} for the surface. **[10]**

Question 5. In the surface of revolution S of Question 4, let

$$f(u) = \frac{2}{3}(1-u)^{3/2}$$

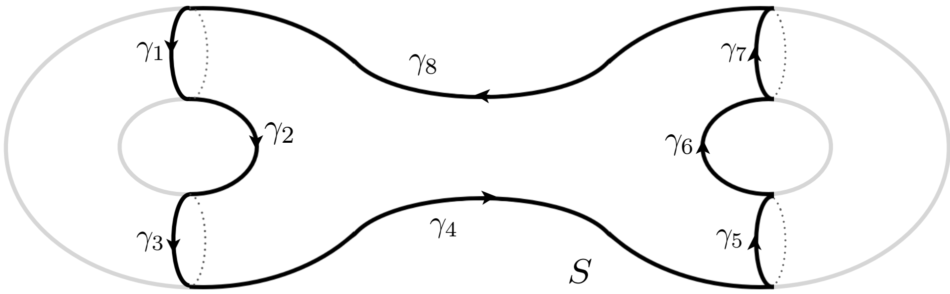
be the axial distance function, $\gamma(t) = \sigma(u(t), v(t))$ a curve in S and $\Omega = f^2\dot{v}$.

(a) Show using F_I from Question 4 that γ is unit speed if and only if $\dot{u} = \pm\sqrt{1 - \frac{\Omega^2}{f^2}}$. **[3]**

(b) State a condition on Ω for a unit speed γ to be a geodesic. *[You are not required to prove anything.]* **[2]**

(c) Suppose that γ is a unit speed geodesic starting at an initial point where u, v are close to 0 and $\dot{v} = 1$, $\dot{u} > 0$. Sketch how the geodesic proceeds on the surface, justifying your answer. **[8]**

Question 6. Let $\gamma = \cup_{i=1}^{i=8}\gamma_i$ be the curvilinear polygon as shown on the 2-holed torus S.



Here $\gamma_2, \gamma_4, \gamma_6, \gamma_8$ all lie in a plane that cuts S in half. The semicircles $\gamma_1, \gamma_3, \gamma_5, \gamma_7$ come up from this plane and coincide with parts of meridians on a standard torus.

(a) Are γ_1, γ_3, γ_5, γ_7 geodesics? Justify your answer. **[2]**

(b) Are $\gamma_2, \gamma_4, \gamma_6, \gamma_8$ geodesics? Justify your answer. **[5]**

(c) Use the Gauss-Bonnet theorem for curvilinear polygons to deduce that

$$\int_S K_G \mathrm{d}A = -4\pi$$

where K_G is the Gauss curvature. You may assume that $\int K_G \mathrm{d}A = 0$ over the half-torus on the left and the half-torus on the right. **[8]**

End of Paper.