Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH5109: Geometry II: Knots and Surfaces

Duration: 2 hours

Date and time: 20th May 2014, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): S. Majid

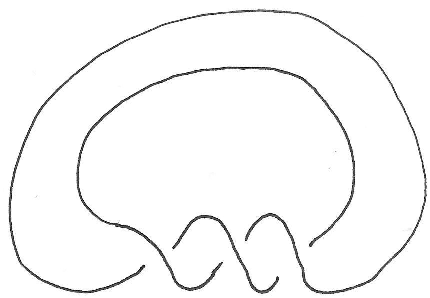
Question 1

(a) Give an example of a knot that is *tricolourable*, justifying your answer. [4]

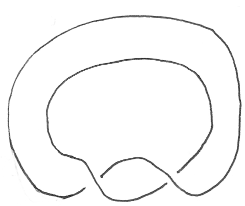
(b) Use tricolourability to prove that the knot you wrote down in part (a) is *not* equal to the trivial circle knot, explaining your logic. [4]

Question 2

(a) Compute the *Kauffman bracket* $B(x)$ of the knot diagram



You may assume without proof that

$B($  $) = -x^4 - x^{-4}$

and you may refer without proof to the behaviour of B under a Reidemeister move of type I. [6]

(b) Compute the *Jones invariant* $J(t)$ of the knot depicted in part (a). [5]

(c) Use results about $J(t)$ from lectures to prove that the knot in part (a) is *chiral*. [3]

Question 3

(a) State the Serret-Frenet equations for the orthonormal triple $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ associated to a curve γ in $\mathbb{R}^3$, including definitions of $\mathbf{t}, \mathbf{n}, \mathbf{b}$ in terms of the curve. [5]

(b) Using part (a), prove that if a regular curve has zero torsion then the curve lies in a plane. You may assume that the curve has non-vanishing curvature and you may wish to look at $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{b} \cdot \gamma)$. [6]

Question 4

(a) Sketch the closed curve $\gamma(t) = (\cos t, e^{\sin t})$ in the $x-y$-plane. (Hint: compared to a circle, the y-coordinate is exponentiated). [5]

(b) State the accumulated angular change in direction on going around the curve in part (a). [2]

(c) Compute the signed curvature K_S of the curve in part (a). [6]

(d) Combining the above with a result from lectures, or otherwise, prove that

$$\int_0^{2\pi} \frac{1 - \sin t \cos^2 t}{\sin^2 t + e^{2\sin t} \cos^2 t} e^{\sin t} \mathrm{d}t = 2\pi.$$

[5]

Question 5

(a) Sketch the surface of revolution with surface patch

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u)), \quad 0 < u < 1,\ 0 < v < 2\pi,$$

$$f(u) = u^2, \quad g(u) = u\sqrt{1 - u^2} + \arcsin u,$$

where $0 < \arcsin u < \frac{\pi}{2}$. You may assume without proof that $g'(u) = 2\sqrt{1 - u^2}$ and hence that $g(u)$ increases with u. [10]

(b) Compute the 1st and 2nd fundamental forms F_I, F_{II} for the surface in part (a). [10]

Question 6

(a) Show that a unit speed geodesic $\gamma(t) = \sigma(u(t), v(t))$ in the surface of revolution in Question 5 obeys

$$\dot{u}^2 = \frac{1}{4}\left(1 - \frac{\Omega^2}{u^4}\right).$$

You may assume that $\Omega = f^2\dot{v}$ is constant along the geodesic, where f is as in Question 5, and you may wish to use F_I obtained there. [6]

(b) Suppose in part (a) that a geodesic has $\Omega = \frac{1}{4}$ and $\dot{u} < 0$, $\dot{v} > 0$ at an initial point where u is close to 1 and v is close to 0. Sketch what the geodesic looks like on the surface and calculate the closest horizontal distance it comes to from the z-axis. [5]

Question 7

(a) State the Gauss-Bonnet theorem for a simple closed curve γ in an orientable surface S. [5]

(b) Let $\gamma(t) = (a\cos t, a\sin t, -\sqrt{1-a^2})$ be a horizontal circle of radius a, $0 < a \leqslant 1$, in the lower part of a unit sphere. Indicate γ and $\text{int}(\gamma)$ on a sketch and show that

$$\text{Area}(\text{int}(\gamma)) = (1 - \sqrt{1-a^2})2\pi.$$

You may assume that $\sqrt{\det(F_I)} = \cos u$ in the standard surface patch used in lectures (where $-\frac{\pi}{2} < u < \frac{\pi}{2}$ is the angle of lift from the $x-y$ plane and $\sin u = -\sqrt{1-a^2}$ for a point on γ). [6]

(c) Use parts (a) and (b) to deduce the value of the geodesic curvature K_g as a function of a. You may assume that the sphere has Gauss curvature $K_G = 1$ and that K_g has the same value at all points of γ. [5]

(d) For which value of a is γ in part (b) a geodesic? Justify your answer. [2]

End of Paper