Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH5105: Differential and Integral Analysis

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **2 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: Huy T. Nguyen, Mira Shamis

Question 1 [25 marks].

(a) Let $q : (a, b) \to \mathbb{R}$ be a real valued function. State the definition for q to be **differentiable** at a point $z \in (a, b)$. Give a geometric explanation of what it means to differentiable. **[5]**

(b) Consider the following formula, g, given by

$$g(x) = \frac{1}{\sqrt{|x|}}.$$

Give the domain and range of this function. Using the definition of the derivative, compute the derivative of g, wherever it exists. Where it does not exist, explain why. **[5]**

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Is f differentiable on $\mathbb{R}$? If yes, prove it. If not, show that it is not. **[6]**

(d) Let $f(x)$ be twice differentiable in the interval (a, b) and suppose that $f''(x) \leq 0$ for every value of x. If x_0 is any point in the interval, the tangent line at x_0 is given by $y_0(x) = f(x_0) + f'(x_0)(x - x_0)$. Show that f always lies below its tangent line, that is $f(x) - y_0(x) \leq 0$ for any x. **[9]**

Question 2 [25 marks].

(a) Prove that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, 1]$ for any $0 < a < 1$. **[5]**

(b) Let $\{g_n\}_{n=1}^{\infty}$ where $g_n : I \to \mathbb{R}$ be a sequence of functions from an interval I. State the definition for $\{g_n\}_{n=1}^{\infty}$ to be a **uniformly convergent sequence of functions**. Explain how uniformly convergent sequences differ from pointwise convergent sequences. **[5]**

(c) Consider the sequence of functions

$$f_n(x) = \begin{cases} \frac{1}{1+x^{-n}}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

(i) Let $f(x) = \lim_{n\to\infty} f_n(x)$. Compute $f(x)$. **[5]**

(ii) Does f_n converge to f uniformly on $[0, a]$, for $a < 1$? Justify your answer. **[5]**

(iii) Does f_n converge to f uniformly on $[0, 1]$? Justify your answer. **[5]**

Question 3 [25 marks].

(a) Let $f : (a, b) \to \mathbb{R}$ be a function that is infinitely differentiable at $x = 0$. State Taylor's formula for f about 0. Explain the relationship between Taylor's formula and the Mean Value Theorem. **[5]**

(b) Let $f(x) = \tanh(x)$ for $x \in \mathbb{R}$ and let $g(y)$ denote the inverse of f. Compute the derivative of $g(y)$ in terms of y. **[5]**

(c) Let $h : \mathbb{R} \backslash \{\pm a\} \to \mathbb{R}$ be the function given by

$$h(x) = \frac{1}{a^2 - x^2}.$$

(i) Using any correct method, find the Taylor series of h about $x = 0$ and find its radius of convergence and interval of convergence. What happens to the Taylor series and the function h outside the radius of convergence? **[7]**

(ii) Compute the derivative of h. **[2]**

(iii) Hence, using (ii) or otherwise, find the Taylor series, radius of convergence and interval of convergence for

$$g(x) = \frac{2x}{a^2 - x^2}$$

about $x = 0$. **[6]**

Question 4 [25 marks].

(a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. State the definition for the **Upper sum and Lower sum** of f. What is the relationship between the Upper and Lower Sum and the integral of f? **[5]**

(b) Let $f_n(x) = \frac{1}{n}x^n \sin^{2022}(x), x \in [0,1]$. Compute $\lim_{n\to\infty} f_n(x)$ and compute

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx.$$ **[5]**

(c) State the **Mean Value Theorem for Integrals**. From the course, state a theorem the Mean Value Theorem for Integrals is used to prove. **[5]**

(d) Let $f : [a,b] \to \mathbb{R}$ denote a bounded function and let F, G be antiderivatives of f. What is the relationship between F and G? Prove any claim that you make. **[5]**

(e) Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-7}^{x^3} f(u)du, \quad u \in \mathbb{R}.$$

Show that F is differentiable on $\mathbb{R}$ and compute its derivative F'. **[5]**

End of Paper.