Queen Mary
University of London

MTH5104: Convergence and Continuity

Duration: 2 hours

Date and time: 20 May 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): R. Buzano

You may assume any standard properties of the sine, cosine and exponential functions including the fact that they are continuous.

Question 1 (15 marks).

(a) Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to be a *Cauchy sequence*. **[3]**

(b) For each of the following sequences, state whether the sequence converges to a limit in $\mathbb{R}$, and, if so, find the limit. Give reasons for your answers. (*You may use any results from the course provided you state clearly which result you are using.*)

(i) $x_n = (-1)^n \left(\dfrac{n^2+1}{2n^2-1} \right)$; **[4]**

(ii) $y_n = (-1)^n \left(\dfrac{n^2+1}{2n^3-1} \right)$; **[4]**

(iii) $z_n = \cos\left(\exp\left(\dfrac{n^2+1}{2n^3-1} \right) \right)$; **[4]**

Question 2 (15 marks). Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x^5 - 15x^4 - 10x^2 + 20.$$

(a) State the Intermediate Value Theorem. **[3]**

(b) Prove that the equation $f(x) = 0$ has at least two different real solutions in the interval $[-1, +1]$. **[6]**

(c) Prove that the equation $f(x) = 0$ has a real solution with $x \geqslant 1$. **[6]**

Question 3 (24 marks).

(a) Define (using quantifier expressions) what it means to say that a sequence $(x_n)_{n=1}^{\infty}$ *tends to infinity*. **[3]**

(b) Prove directly from the definition that the sequence $(x_n)_{n=1}^{\infty}$, defined by $x_n = \sqrt[3]{n} + 1$ for all $n \in \mathbb{N}$, tends to infinity. **[5]**

(c) Define (using quantifier expressions) what it means to say that a sequence $(y_n)_{n=1}^{\infty}$ *converges to zero*. **[3]**

(d) Prove directly from the definition that the sequence $(y_n)_{n=1}^{\infty}$, defined by $y_n = \frac{1}{\sqrt[3]{n}+1}$ for all $n \in \mathbb{N}$, converges to zero. **[5]**

(e) Prove that whenever $(x_n)_{n=1}^{\infty}$ is a sequence with $x_n \neq 0$ for all n and which tends to infinity, then the sequence $(y_n)_{n=1}^{\infty}$, defined by $y_n = \frac{1}{x_n}$ for all $n \in \mathbb{N}$, converges to zero. **[8]**

Question 4 (15 marks).

(a) Which of the following series converge? Justify your answers.

$$(i)\ \sum_{k=1}^{\infty}\frac{1}{k^3+1}, \qquad (ii)\ \sum_{k=1}^{\infty}\left(\frac{1}{k^3}-\frac{1}{k!}\right), \qquad (iii)\ \sum_{k=1}^{\infty}\left(\frac{1}{k^3}+\frac{1}{k}\right).$$

(*You may use any results from the course provided you state clearly which result you are using.*) **[9]**

(b) For which $x \in \mathbb{R}$ does the series $\sum_{k=1}^{\infty}\frac{(-x)^k}{k}$ converge? Justify your answer. **[6]**

Question 5 (19 marks). Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Define (using quantifier expressions) what it means to say that f is *continuous at a point* $a \in \mathbb{R}$. **[3]**

(b) Prove: If f is continuous at $a \in \mathbb{R}$ and $(x_n)_{n=1}^{\infty}$ converges to a, then $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$. **[10]**

(c) Prove that the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} \cos(x) & \text{if } x \in \mathbb{Q}, \\ \sin(x) & \text{if } x \notin \mathbb{Q}, \end{cases}$$

is *not* continuous at $a = 0$. **[6]**

Question 6 (12 marks). Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function. We say that $f(x)$ tends to zero as x tends to infinity if

$$\forall \varepsilon > 0\ \exists K \in \mathbb{R}\ \forall x > K\ :\ |f(x)| < \varepsilon.$$

(a) Prove that the function $f(x) = \frac{\sin(x)}{x^2}$ tends to zero as x tends to infinity. **[6]**

(b) Does the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z} \\ 0 & \text{if } x \notin \mathbb{Z} \end{cases}$$

tend to zero as x tends to infinity? Justify your answer. **[6]**

End of Paper.