Queen Mary
University of London

# B. Sc. Examination by course unit 2014

# MTH5104: Convergence and Continuity

## Duration: 2 hours

## Date and time: 29 May 2014, 10:00–12:00

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**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt all questions. Marks awarded are shown next to the questions.**

**Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.**

**Complete all rough workings in the answer book and cross through any work which is not to be assessed.**

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**Examiner(s): R. Müller**

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**You may assume any standard properties of the sine, cosine and exponential functions including that they are continuous.**

**Question 1** (13 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^2 + 2x + 2$.

(a) State the definition that the function $f$ is continuous at a point $a \in \mathbb{R}$. [3]

(b) Using only this definition of continuity (and not any theorems proved in the course), prove that $f$ is continuous at all points $a \in \mathbb{R}$. [10]

**Question 2** (15 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 2x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that $f$ is continuous at the point $a = 0$. [5]

(b) Is $f$ continuous at any other points in $\mathbb{R}$? Prove your assertion. [10]

**Question 3** (18 marks)

(a) Define what it means to say that $(x_n)_{n=1}^{\infty}$ *converges to zero*, and prove that if $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ both converge to zero, then so does $(x_n \cdot y_n)_{n=1}^{\infty}$. [6]

(b) For each of the following sequences, state whether the sequence converges to a limit in $\mathbb{R}$, and, if so, find the limit. Give reasons for your answers. *(You may use any results from the course provided you state clearly which result you are using.)*

(i) $x_n = (-1)^n \left(\dfrac{n-1}{n+2}\right)$; [4]

(ii) $x_n = \sin\left(\frac{\pi}{2} \cdot \exp\left(\dfrac{1}{n}\right)\right)$; [4]

(iii) $x_n = \cos\left(\dfrac{n^2+1}{n^3+2}\right)$. [4]

**Question 4** (14 marks) Let the sequence $(x_k)_{k=1}^{\infty}$ be defined inductively by $x_1 = 14$, $x_{n+1} = 6 + \sqrt{x_n - 5}$.

(a) Compute $x_2$, $x_3$ and $x_4$. [2]

(b) Prove that 7 is a lower bound for $(x_k)_{k=1}^{\infty}$. [4]

(c) Prove that $(x_k)_{k=1}^{\infty}$ is strictly decreasing. [4]

(d) Deduce that $(x_k)_{k=1}^{\infty}$ converges and compute the limit. [4]

**Question 5** (10 marks) Let $(x_k)_{k=1}^{\infty}$ be a sequence. For each of the following statements state whether it is true or false. Justification is *not* required.

(a) If $a < x_k < b$ for all $k \in \mathbb{N}$, then $(x_k)_{k=1}^{\infty}$ has an accumulation point in $(a, b)$.

(b) If $a \leqslant x_k \leqslant b$ for all $k \in \mathbb{N}$, then $(x_k)_{k=1}^{\infty}$ has an accumulation point in $[a, b]$.

(c) If $(x_k)_{k=1}^{\infty}$ converges to zero, then $\sum_{k=1}^{\infty} x_k$ exists.

(d) If $\sum_{k=1}^{\infty} x_k$ exists, then $(x_k)_{k=1}^{\infty}$ converges to zero.

(e) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} (-1)^k x_k$ exists.

(f) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} \frac{x_k}{k}$ exists.

[10]

**Question 6** (13 marks)

(a) Given a sequence $(x_k)_{k=1}^{\infty}$ and a real number $S$, what does it mean to say that the sum $\displaystyle\sum_{k=1}^{\infty} x_k$ exists and equals $S$? [3]

(b) Which of the following sums exist? Briefly justify your answers.

$$(i)\ \sum_{k=1}^{\infty} \frac{1}{k^4}, \qquad (ii)\ \sum_{k=1}^{\infty} \frac{1}{2^{2k}}, \qquad (iii)\ \sum_{k=1}^{\infty} \frac{1}{2k}$$

*(You may use any results from the course provided you state clearly which result you are using.)* [6]

(c) Does the sum

$$\sum_{k=1}^{\infty} \left( \frac{1}{k^4} + \frac{1}{2^{2k}} - \frac{1}{2k} \right)$$

exist? Prove your assertion. [4]

**Question 7** (17 marks)

(a) State the Intermediate Value Theorem. [3]

(b) Show that there exist at least two different real solutions to the equation $(\sin x)^2 = (\cos x)^4$ in the interval $[0, \pi]$. [7]

(c) Suppose that $f\colon [0,1] \to [0,1]$ is a continuous function. By considering a suitable function $g$, or otherwise, prove that there must exist a point $c \in [0,1]$ with $f(c) = \sqrt[3]{c}$. [7]

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**End of Paper**