Queen Mary University of London

Main Examination period 2023 – May/June – Semester B

MTH5103: Complex Variables

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: M. Shamis, M. Farber

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Question 1 [20 marks].

(a) Find all the solutions of the following equation

$$3z + (2 - i)\bar{z} = 7 + i$$

in Cartesian form. Justify all of your steps. [**10**]

(b) Compute the following number

$$\left(1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6.$$

Justify all of your steps. [**10**]

Question 2 [20 marks].

(a) Using the Ratio Test, or otherwise, determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^n}{(n!)^5}(z - 2i)^n.$$

State the test that you are using and justify all of your steps.

[**10**]

(b) Using the Root Test, or otherwise, determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n^{5n}}{(4n^4 + (1+i)n^3 - 4in^5)^n}(z - 3i)^n.$$

State the test that you are using and justify all of your steps. [**10**]

Question 3 [20 marks].

(a) Find the coefficients a_n and b_n of the Laurent series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$$

of the function $f(z) = z^{15}\sin\left(\frac{1}{z^5}\right)$ valid on the set $\{z \in \mathbb{C} : 0 < |z| < 10\}$, where $z_0 = 0$. Justify all of your steps. [**15**]

(b) Determine the residue of the function $f(z)$ from part (a) at the point $z = 0$. Justify all of your steps. [**5**]

Question 4 [20 marks].

(a) Express $\text{Log}\left(\frac{3+4i}{2+i}\right)$ in Cartesian form $(x + iy)$. Justify all of your steps. [**5**]

(b) Verify that the following function is harmonic

$$u(x, y) = -e^{y} \sin x - 2e^{-x} \sin y \ .$$

Find a harmonic conjugate v of u, namely a function v such that the function $u(x, y) + iv(x, y)$ is an analytic function, such that $v(0, 0) = 4$. Justify all of your steps. [**15**]

Question 5 [20 marks].

(a) Find all singularities of the function

$$f(z) = \frac{\sin \frac{1}{z} \cos(z - \pi)}{\left(z - \frac{3\pi}{2}\right)(z + 4i)^2},$$

and determine the nature of each of these singularities (e.g. removable singularity, pole, essential singularity). Justify all of your steps. [**10**]

(b) Using the Residue Theorem, or otherwise, compute

$$\int_C \frac{z - 7}{z^2 + z - 2}\, dz,$$

where C is the positively oriented circle of radius 4 centered at $z = 0$. **State all the theorems that you are using and justify all of your steps**. [**10**]

End of Paper.