Queen Mary University of London

Main Examination period 2020 – January – Semester A

MTH5102: Calculus III (resit paper)

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: A. Saha

Section A - multiple choice & short questions

This section contains multiple-choice questions and short-answer questions. Use the back of the answer book for rough work. Questions left unanswered and incorrect answers both get zero marks. Each multiple-choice question has exactly one correct answer.

Question 1 [4 marks]. Consider the surface described implicitly by $f(x, y, z) = 1$, where $f(x, y, z) = x^2 + y^3 + z^4$. The equation of the plane tangent to the surface at the point with coordinates $(0, 1, 0)$ is **[4]**

☐ $y = 1$
☐ $\mathbf{r}(t) = (0, 0, 0) + t(0, 3, 0)$
☐ $2x + 3y + 4z = 3$
☐ none of the above

Question 2 [3 marks]. $x^2 + y^2 + 3z^2 = 16$ is the implicit equation of **[3]**

☐ a circle
☐ an ellipsoid
☐ a hyperboloid
☐ a paraboloid

Question 3 [3 marks]. Consider the surface in $\mathbb{R}^3$ given in parametric form by $\mathbf{r}(u, v)$. Then $\left(2\frac{\partial \mathbf{r}}{\partial u} + 3\frac{\partial \mathbf{r}}{\partial v}\right)\Big|_{(u_0,v_0)}$ is **[3]**

☐ a vector tangent to the surface at the point $\mathbf{r}(u_0, v_0)$
☐ the equation of the line passing through $\mathbf{r}(u_0, v_0)$ and normal to the surface
☐ a vector normal to the surface at the point $\mathbf{r}(u_0, v_0)$
☐ none of the above

Question 4 [5 marks]. Consider the curve in $\mathbb{R}^2$ given in **polar coordinate** form by the equation $r = \theta$. Make a rough sketch of the curve from $\theta = 0$ to $\theta = 2\pi$. [**5**]

Question 5 [3 marks]. $\mathbf{r}(t) = (t, t, 0)$ gives the parametric equation of [**3**]

☐ a circle centred at $(1, 1, 1)$
☐ a circle centred at $(1, 0, 0)$
☐ a line passing through the origin
☐ none of the above

Question 6 [2 marks]. Suppose that the vector field **F** represents the velocity of some fluid at each point in space. If the curl of **F** is equal to the zero vector at some point P, it tells us that [**2**]

☐ more fluid is flowing out of P than flowing in, i.e., P is a net source for the fluid
☐ more fluid is flowing into P than flowing out, i.e., P is a net sink for the fluid
☐ the fluid is rotating around P in a clockwise direction
☐ none of the above

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Section B

Respond to the following questions using the space provided for that purpose.

Question 7 [25 marks].

(a) Let U be a scalar field. Write the expression for $\nabla^2 U$ in three dimensional Cartesian coordinates. **[3]**

(b) State Stokes's theorem. **[5]**

(c) Show, using Stokes's theorem, that the line integral of an irrotational vector field around a closed curve is null. **[6]**

(d) Let $\mathbf{F} = (2x, -2y, 1)$. Prove that for this case $\mathbf{F}$ is conservative and compute a scalar potential ϕ for $\mathbf{F}$. **[9]**

(e) State (without proof) whether the following is (i) a scalar field (ii) a vector field or (iii) not a valid expression:

$$\mathbf{F} - \nabla \times \mathbf{F} + (\nabla \cdot ((\nabla \times \mathbf{F})U)).$$

[2]

Write your solution to Question 7 above

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Question 8 [15 marks].
Let $\mathcal{C}$ be the curve in $\mathbf{R}^3$ whose parametric equation reads $\mathbf{r}(t) = (4t, t, t^2)$, and consider the points $A = (1, 1/4, 1/16), B = (4, 1, 1), C = (2, 0, 0)$.

(a) Which points from $\{A, B, C\}$ belong to the curve $\mathcal{C}$? Justify your answer. **[3]**

(b) State a parametric equation for the tangent line to $\mathcal{C}$ at point A. **[5]**

(c) Consider the vector field $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} - 4y\mathbf{k}$. Calculate the line integral of $\mathbf{F}$ over $\mathcal{C}$, between the origin and A. **[7]**

Write your solution to Question 8 above

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Question 9 [15 marks].

(a) State the definition of a **solenoidal field**. Prove that if **A** is a vector field in $\mathbb{R}^3$, then the vector field $\nabla \times \mathbf{A}$ is solenoidal. **[7]**

(b) Let S be the surface given by $\{x^2 + y^2 + z^2 = 4,\ 0 \leq z \leq 2\}$. Is S a closed oriented surface? **[2]**

(c) The vector field **F** is given by **F** = z **i**. Using the divergence theorem, or otherwise, calculate the flux of **F** across S. **[6]**

[Hint: For the last part above, you may find it useful to recall that a sphere of radius a has volume equal to $\frac{4}{3}\pi a^3$.]

Write your solution to Question 9 above

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Question 10 [25 marks]. Let $\Phi(x, y)$ be a scalar field that satisfies Laplace's equation $\nabla^2\Phi = 0$ inside the rectangle D given by $0 \leq x \leq 2,\ 0 \leq y \leq 1$. The boundary conditions are: $\Phi(x, 0) = x/2$, $\Phi(0, y) = 0$, $\Phi(x, 1) = 0$, $\Phi(2, y) = 1 - y^2$.

(a) Write down the values of $\Phi(x, y)$ at the four corners of D. **[3]**

(b) Find coefficients $\alpha, \beta, \gamma, \delta$ such that the values of the function $h(x, y) = \alpha + \beta x + \gamma y + \delta xy$ and that of the function $\Phi(x, y)$ match at the four corners of D. **[4]**

(c) Consider the function $f(x, y) = \Phi(x, y) - h(x, y)$. Find the Fourier sine series for $f(2, y)$. [Hint: Use integration by parts] **[9]**

(d) Find $\Phi(x, y)$. [Hint: First find $f(x, y)$] **[9]**

[For the last two parts above, you may find it useful to look at the appendix after the end of this paper.]

Write your solution to Question 10 above

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End of Paper – An appendix of 2 pages follows.

Some trigonometric identities

$$
\begin{aligned}
\sin^2 A + \cos^2 A &= 1 \\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin(3A) &= -4\sin^3(A) + 3\sin(A) \\
\cos(3A) &= 4\cos^3(A) - 3\cos(A)
\end{aligned}
$$

For integers m, n not both equal to 0, and a real number a, we have

$$\frac{2}{a}\int_0^a \sin(m\pi x/a)\sin(n\pi x/a) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$\frac{2}{a}\int_0^a \cos(m\pi x/a)\cos(n\pi x/a) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$\int_{-a}^{a} \sin(m\pi x/a)\cos(n\pi x/a) = 0.$$

Cylindrical coordinates (covered in class)

The cylindrical coordinates (ρ, ϕ, z) are related to the usual Cartesian coordinates (x, y, z) by $x = \rho\cos\phi$, $y = \rho\sin\phi$. The corresponding unit vectors in cylindrical coordinates $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$ are orthonormal. The relation between the unit vectors in Cartesian coordinates and the unit vectors in cylindrical coordinates can be expressed by the equation

$$\begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$

In cylindrical coordinates, the area element $\mathbf{dS}$ is given by $\rho\,\mathbf{e}_\rho\,d\phi\,dz$ while the volume element dV is given by $\rho\,d\rho\,d\phi\,dz$.

Dirichlet's problem (Variant of boundary Case I)

Consider the rectangle $D = \{0 \leq x \leq a, 0 \leq y \leq b\}$. Suppose we want to find a smooth real-valued function Φ on D satisfying the following properties:

(a) $\nabla^2\Phi = 0$,

(b) $\Phi = 0$ on the three sides $x = 0$, $y = 0$, $y = b$,

(c) $\Phi = g(y)$ on the side $x = a$ where $g(y)$ is a smooth real-valued function satisfying $g(0) = g(b) = 0$.

Then the unique solution to the above problem is

$$\Phi(x, y) = \sum_{n=1}^{\infty} \frac{E_n}{\sinh(n\pi a/b)}\sinh(n\pi x/b)\sin(n\pi y/b)$$

where the coefficients E_n are the (sine) Fourier coefficients for $g(y)$, i.e., for all $0 \le y \le b$,

$$g(y) = \sum_{n=1}^{\infty} E_n \sin(n\pi y/b).$$

The coefficients E_n can be directly found via the formula

$$E_m = \frac{2}{b} \int_0^b g(y) \sin(n\pi y/b) dy.$$

End of Appendix.