Queen Mary University of London

Main Examination period 2018

MTH5102: Calculus III

Duration: 2 hours

Student number | | | | | | | | | | **Desk number** | | | |

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Write your solutions in the spaces provided in this exam paper. If you need more paper, ask an invigilator for an additional booklet and attach it to this paper at the end of the exam.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: A. Saha

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Do not write on it.

Question	Mark	Comments
1	/ 4	
2	/ 3	
3	/ 3	
4	/ 5	
5	/ 3	
6	/ 2	
7	/ 25	
8	/ 15	
9	/ 20	
10	/ 20	
Total		

Section A - multiple choice & short questions

This section contains multiple-choice questions and short-answer questions. Use the back of the answer book for rough work. Questions left unanswered and incorrect answers both get zero marks. Each multiple-choice question has exactly one correct answer.

Question 1. [4 marks] Consider the curve in $\mathbb{R}^3$ described in parametric form by $\mathbf{r}(t) = (f(t), g(t), h(t))$. The arc-length along the curve between the points $\mathbf{r}(t_1)$ and $\mathbf{r}(t_2)$ equals **[4]**

- ☐ $\int_{t_1}^{t_2} \sqrt{f(t) + g(t) + h(t)}\, dt$
- ☐ $\sqrt{(f(t_2) - f(t_1))^2 + (g(t_2) - g(t_1))^2 + (h(t_2) - h(t_1))^2}$
- ☐ $\int_{t_1}^{t_2} \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$
- ☐ $|\mathbf{r}(t_2) - \mathbf{r}(t_1)|$

Question 2. [3 marks] The gradient of a scalar field can tell us **[3]**

- ☐ the direction in which the scalar field increases most quickly at each point
- ☐ whether the field is rotating or not and what is the direction and speed of rotation
- ☐ whether the field is locally a sink or a source
- ☐ none of the above

Question 3. [3 marks] Consider the surface in $\mathbb{R}^3$ given in parametric form by $\mathbf{r}(u, v)$. Then $\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\Big|_{(u_0,v_0)}$ is **[3]**

- ☐ a tangent vector to the surface at the point $\mathbf{r}(u_0, v_0)$
- ☐ the tangent plane to the surface at the point $\mathbf{r}(u_0, v_0)$
- ☐ a vector normal to the surface at the point $\mathbf{r}(u_0, v_0)$
- ☐ none of the above

Question 4. [5 marks] Consider the curve in $\mathbb{R}^2$ given in **polar coordinate** form by the equation $r = |\cos(\theta)|$. Make a rough sketch of the curve from $\theta = 0$ to $\theta = 2\pi$. **[5]**

Question 5. [3 marks] $\mathbf{r}(t) = (\cos(t), \sin(t), 1)$ gives the parametric equation of **[3]**

- ☐ a circle centred at $(1, 1, 1)$
- ☐ a circle centred at $(1, 0, 0)$
- ☐ a circle centred at $(0, 0, 1)$
- ☐ a spiral

Question 6. [2 marks] Exactly one of the items below is **not** a vector field. Which one? **[2]**

- ☐ the gradient of the temperature at each point in space
- ☐ the velocity field of a moving fluid
- ☐ the gravitational field generated by any massive object
- ☐ the humidity at each point in space

Section B

Respond to the following questions using the space provided for that purpose.

Question 7. [25 marks]
Let the scalar field U be given by $U(x, y, z) = x^2 + 8x + 4y^2 + 9z^2$, and the vector field $\mathbf{F}$ be given by $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$.

(a) Calculate the divergence and curl of $\mathbf{F}$. **[3+4]**

(b) Briefly explain why the vector field $\mathbf{F}$ is conservative, and find a scalar potential for $\mathbf{F}$. **[2+6]**

(c) Let S be the closed surface defined by $U(x, y, z) = 0$. Is S an ellipsoid, a paraboloid, or a hyperboloid? **[2]**

(d) With S as above, compute $\int_S \mathbf{F} \cdot \mathbf{d}\boldsymbol{S}$. [Hint: Use the divergence theorem.] **[5]**

(e) For each of the following expressions, state (without proof) whether it is (i) a scalar field (ii) a vector field or (iii) not a valid expression:

$$(\nabla \times \mathbf{F})U \; ; \quad (\nabla \cdot \mathbf{F})U \; ; \quad \nabla \times (\nabla \cdot \mathbf{F}) \; .$$

[3]

Write your solution to Question 7 above

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Question 8. [15 marks]

(a) State without proof Stokes's theorem (don't forget to include any required conditions). **[5]**

(b) The vector field **F** is given by $\mathbf{F} = (4z - 2y)\mathbf{i} + (2x - 3z)\mathbf{j} + (x + z^2)\mathbf{k}$. Using Stokes's theorem, or otherwise, evaluate the line integral of **F** around the unit circle $x^2 + z^2 = 1, y = 1$, traversed anticlockwise around the y-axis, starting and finishing at the point (1, 1, 0). **[10]**

Write your solution to Question 8 above

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Question 9. [20 marks]

(a) Let f, g be differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$ and let **F** be the vector field given by $\mathbf{F}(x, y, z) = (f(x, y), g(x, y), 0)$.
Prove that **F** is curl-free if and only if $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. **[7]**

(b) Let S be the surface given by $\{x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$. Is S a closed surface? Justify your answer. **[3]**

(c) The vector field **F** is given by **F** = y **i**. Express **F** in *cylindrical* coordinates. **[4]**

(d) Let **F** and S be as in the previous two parts. Calculate the flux of **F** across S. **[6]**

[For the last two parts above, you may find it useful to look at the appendix after the end of this paper.]

Write your solution to Question 9 above

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Question 10. [20 marks] Let $\Phi(x, y)$ be a scalar field that satisfies Laplace's equation $\nabla^2\Phi = 0$ inside the rectangle D given by $0 \le x \le \pi,\ 0 \le y \le 1$. The boundary conditions are: $\Phi(x, 0) = x$, $\Phi(0, y) = 0$, $\Phi(x, 1) = \sin^3(x)$, $\Phi(\pi, y) = \pi - \pi y$.

(a) Write down the values of $\Phi(x, y)$ at the four corners of D. **[3]**

(b) Find coefficients $\alpha, \beta, \gamma, \delta$ such that the values of the function $h(x, y) = \alpha + \beta x + \gamma y + \delta xy$ and that of the function $\Phi(x, y)$ match at the four corners of D. **[4]**

(c) Express the function $\sin^3(x)$ as a Fourier series. **[4]**

(d) Find $\Phi(x, y)$. [Hint: First find $\Phi(x, y) - h(x, y)$] **[9]**

[For the last two parts above, you may find it useful to look at the appendix after the end of this paper.]

Write your solution to Question 10 above

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End of Paper – An appendix of 2 pages follows.

Some trigonometric identities

$$
\begin{aligned}
\sin^2 A + \cos^2 A &= 1\\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B\\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B\\
\sin(3A) &= -4\sin^3(A) + 3\sin(A)\\
\cos(3A) &= 4\cos^3(A) - 3\cos(A)
\end{aligned}
$$

For integers m, n not both equal to 0, and a real number a, we have

$$\frac{2}{a}\int_0^a \sin(m\pi x/a)\sin(n\pi x/a) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$\frac{2}{a}\int_0^a \cos(m\pi x/a)\cos(n\pi x/a) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$$\int_{-a}^a \sin(m\pi x/a)\cos(n\pi x/a) = 0.$$

Cylindrical coordinates (covered in class)

The cylindrical coordinates (ρ, ϕ, z) are related to the usual Cartesian coordinates (x, y, z) by $x = \rho\cos\phi$, $y = \rho\sin\phi$. The corresponding unit vectors in cylindrical coordinates $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$ are orthonormal. The relation between the unit vectors in Cartesian coordinates and the unit vectors in cylindrical coordinates can be expressed by the equation

$$\begin{pmatrix}\mathbf{e}_\rho\\ \mathbf{e}_\phi\\ \mathbf{e}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix}$$

In cylindrical coordinates, the area element $\mathbf{dS}$ is given by $\rho\,\mathbf{e}_\rho\,d\phi\,dz$ while the volume element dV is given by $\rho\,d\rho\,d\phi\,dz$.

Dirichlet's problem (Boundary Case I)

Consider the rectangle $D = \{0 \le x \le a, 0 \le y \le b\}$. Suppose we want to find a smooth real-valued function Φ on D satisfying the following properties:

(a) $\nabla^2\Phi = 0$,

(b) $\Phi = 0$ on the three sides $x = 0$, $y = 0$, $x = a$,

(c) $\Phi = g(x)$ on the side $y = b$ where $g(x)$ is a smooth real-valued function satisfying $g(0) = g(a) = 0$.

Then the unique solution to the above problem is

$$\Phi(x, y) = \sum_{n=1}^{\infty} \frac{E_n}{\sinh(n\pi b/a)}\sinh(n\pi y/a)\sin(n\pi x/a)$$

where the coefficients E_n are the (sine) Fourier coefficients for $g(x)$, i.e., for all $0 \leq x \leq a$,

$$g(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a).$$

The coefficients E_n can be directly found via the formula

$$E_m = \frac{2}{a} \int_0^a g(x) \sin(n\pi x/a) dx.$$

End of Appendix.