Queen Mary University of London

Main Examination period 2018

MTH5100: Algebraic Structures I

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: S. Majid, L. Soicher

Question 1. [17 marks]

(a) In a ring R, explain why $0 + 0 = 0$ and hence, or otherwise, show that $a0 = 0$ for all $a \in R$. You may refer to any relevant ring axioms from lectures. **[5]**

(b) State without proof the general form of a **subring** of $\mathbb{Z}$. **[4]**

(c) Define what is meant by a **coset** of a subring S of a ring R. **[4]**

(d) Give an example of a subring S of a ring R where both S, R are **infinite** but the number of cosets of S in R is finite and greater than 1. Justify your answer. **[4]**

Question 2. [15 marks]

(a) Define what is meant by an **ideal** I of a ring R. **[3]**

(b) Let $\theta : R \to S$ be a ring homomorphism between rings R, S. Define $\ker(\theta)$ and $\mathrm{im}(\theta)$. **[4]**

(c) Let $\theta : M_2(\mathbb{Z}) \to M_2(\mathbb{Z}_2)$ be given by taking matrix entries mod 2. Explain why θ is a ring homomorphism and show that $\mathrm{im}(\theta) = M_2(\mathbb{Z}_2)$ and that $\ker(\theta) = M_2(2\mathbb{Z})$. **[8]**

Question 3. [17 marks]

(a) Let I be an ideal of a ring R. What is the zero element of the factor ring R/I? Justify your answer. You are not asked to prove the rest of the ring axioms for R/I. **[4]**

(b) Let $R = M_2(\mathbb{Z})$ and $I = M_2(2\mathbb{Z})$.

(i) Explain why I is an ideal of R and show that $R/I \cong M_2(\mathbb{Z}_2)$. (Hint: use your answer to part (c) of Question 2 and a theorem from lectures.) **[5]**

(ii) Let $T = \{I + A \mid A \in R,\ a_{12} \in 2\mathbb{Z}\}$, where a_{12} is the top right entry of A. Which subring of $M_2(\mathbb{Z}_2)$ does T map to under the isomorphism in part (i)? Justify your answer. **[8]**

Question 4. [26 marks]

(a) Let $i = \sqrt{-1}$. Assuming that the **Gaussian integers** $\mathbb{Z}[i] = \{a + b\,i \mid a, b \in \mathbb{Z}\}$ form a unique factorisation domain, factorise $4 \in \mathbb{Z}[i]$ into irreducibles and outline the sense in which your factorisation is unique. You may assume that the units of the ring are $\pm 1, \pm i$. **[8]**

(b) For a commutative ring, define what it means for one element to **divide** another and what is meant by a **gcd** of two elements. **[6]**

(c) For a commutative ring R with identity, define what is meant by the ideals $\langle a \rangle$, $\langle a, b \rangle$ for any fixed elements $a, b \in R$. **[5]**

(d) Let R be a **principal ideal domain** and let $a, b \in R$. Show that $\langle a, b \rangle = \langle d \rangle$ where d is a gcd of a and b. **[7]**

Question 5. [11 marks]

(a) State precisely what it means for a ring R to be a **Euclidean domain**. **[5]**

(b) Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients and d the function that assigns to a non-zero polynomial its degree. By showing that $1 = 2q + r$ has no solution for $q, r \in \mathbb{Z}[x]$ with $r = 0$ or $d(r) < d(2)$, or otherwise, show that d does **not** make $\mathbb{Z}[x]$ into a Euclidean domain. **[6]**

Question 6. [14 marks]

(a) Define what it means for an ideal I of a ring R to be **maximal**. State what this implies for the factor ring R/I in the case where R is commutative and has an identity. **[5]**

(b) Let $\mathbb{F}_2 = \{0, 1\}$ be the field of 2 elements and $\mathbb{F}_2[x]$ the ring of polynomials with coefficients in $\mathbb{F}_2$. Show that $f(x) = 1 + x^2$ is **not** irreducible as an element of $\mathbb{F}_2[x]$. **[5]**

(c) For f in part (b), is $\mathbb{F}_2[x]/\langle f \rangle$ a **field**? Justify your answer. **[4]**

End of Paper.