MTH4108: Probability I

Duration: 2 hours

Date and time: 9th May 2016, 14:30–16:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): F. Vivaldi

In this paper we employ the following standard notation: Ω is the sample space, ω is an outcome, A, B, C are events, A^c is the complement of A, $|A|$ is the cardinality of A, $\mathbb{P}$ is a probability.

Question 1. (BASICS)

(a) The following expressions are meaningless: explain what is the fault [as opposed to writing a corrected expression]. **[4]**

(i) $\mathbb{P}(\varnothing^c) \cap A$

(ii) $\mathbb{P}(A \subset \Omega)$.

(b) Write down in symbols events such that, of the events A, B and C, **[4]**

(i) only A occurs;

(ii) exactly two of them occur.

No justification is necessary.

(c) Prove the following statements, quoting any result you use. **[4+5]**

(i) For all A, we have $\mathbb{P}(A \cap A^c) = 0$.

(ii) The probability that exactly one of the events A or B occurs is equal to $\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$.

Question 2. (SAMPLING)

(a) A fair coin is tossed repeatedly. Let $n \geqslant 2$. What is the probability that a head will occur for the second time on the nth toss? **[6]**

(b) At a round table with eight chairs, four men and four women sit at random. What is the probability that every woman will sit between two men? **[6]**

(c) Four people choose at random one holiday destination among ten destinations. Show that the probability that at least two people will make the same choice is less than 1/2. **[6]**

Justify your calculations.

Question 3. (CONDITIONAL PROBABILITY)

(a) Let Ω be the interval $[0, 1]$, and let the probability of an interval $A \subset \Omega$ be equal to the length of A. Give an example —with justification— of **[9]**

(i) two intervals A and B such that A and B partition Ω;

(ii) two intervals A and B such that A and B are independent;

(iii) two intervals A and B such that $\mathbb{P}(A|B) = 1$.

(b) In a batch of manufactured items, 5% of the items have a fault. A diagnostic test has 90% chance of detecting an item that is faulty, but also 1% chance of giving a false positive when an item really has no fault. Compute the probability that an item which has been diagnosed as being faulty, is, in fact, faulty. Justify your calculations. **[6]**

(c) Prove that if $\mathbb{P}(A) = \mathbb{P}(B) = \frac{2}{3}$, then $\mathbb{P}(A|B) \geqslant \frac{1}{2}$. **[6]**

Question 4. (RANDOM VARIABLES)

(a) Let the sample space Ω be a set of polygons[1]. Give examples of **[4]**

(i) a discrete random variable on Ω;

(ii) a continuous random variable on Ω.

(b) Let X be a discrete random variable. Consider the expression **[4]**

$$\mathbb{P}(X = k).$$

Explain what the short-hand expression '$X = k$' stands for.

(c) A hundred tickets are sold in a lottery in which there is a top prize of £50 and four prizes of £10 each. Each ticket costs £1. Let X be your <u>net</u> gain when you buy one ticket. **[4+6+3]**

(i) Compute the probability mass function of X.

(ii) Define, compute, and draw the cumulative distribution function of X.

(iii) Compute the expectation of X.

Explain what you do.

[1] A polygon is a plane figure bounded by a finite chain of line segments, closing in a loop.

Question 5. (DISTRIBUTIONS)
In every part of this question, a precise and concise exposition is essential for good marks.

(a) Introduce briefly the Poisson distribution. Verify that it is indeed a distribution, and explain —without proof— its connection with the binomial distribution. [**7**]

(b) Assume that, on average, a typographical error is found every 1000 typeset characters. Compute the probability that a 600-character page contains fewer than two errors. [**4**]

(c) Introduce briefly the exponential distribution, explaining its connection with, and deriving it from, the Poisson distribution. [**7**]

(d) Derive expressions for the median and the expectation of the exponential distribution. [**5**]

End of Paper.