Queen Mary
University of London

MTH4103: Geometry I

Duration: 2 hours

Date and time: 13 May 2016, 10:00–12:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): R. Johnson

Question 1. Let A and B be the points in 3-space with position vectors $\mathbf{a} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ respectively. Determine:

(a) the vector $\mathbf{a} - \mathbf{b}$; **[2]**

(b) the (free) vector represented by the bound vector $\overrightarrow{AO}$; **[3]**

(c) $\mathbf{a} \cdot \mathbf{b}$; **[3]**

(d) $\mathbf{a} \times \mathbf{b}$; **[3]**

(e) a unit vector in the direction of $\mathbf{b}$; **[3]**

(f) parametric equations for the line through A and B; **[3]**

(g) a Cartesian equation for the plane through the origin orthogonal to $\mathbf{b}$ (that is with normal vector $\mathbf{b}$). **[3]**

Question 2. Let Π_1 be the plane with Cartesian equation

$$x - y + 3z = 12,$$

Π_2 be the plane with Cartesian equation

$$y - z = 2,$$

and l be the line with Cartesian equation

$$x = y = \frac{z}{2}.$$

(a) Find the intersection of the two planes Π_1 and Π_2. **[6]**

(b) Find the intersection of the line l with the plane Π_1. **[4]**

(c) Write down a Cartesian equation for the plane Π_3 which is parallel to Π_1 and contains the point with coordinates $(1, 1, 1)$. What is the intersection of the three planes Π_1, Π_2 and Π_3? **[4]**

(d) Write down a Cartesian equation for a plane Π_4 which contains the line l and for which the intersection of the three planes Π_1, Π_2 and Π_4 is empty. **[4]**

(e) Say whether the following statement is true or false, justifying your answer: The intersection of three planes is empty if and only if two of them are parallel. **[2]**

Question 3. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ be vectors, and α and β be real numbers.

(a) State (without proof) a formula for the scalar product $\mathbf{u} \cdot \mathbf{v}$ in terms of coordinates. **[5]**

(b) Define precisely what it means for $\mathbf{u}$ and $\mathbf{v}$ to be **orthogonal**. **[5]**

(c) Use the formula of part (a) to prove that **[6]**

$$(\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) + \beta(\mathbf{v} \cdot \mathbf{w}).$$

(d) Determine, with justification, a necessary and sufficient condition on $\mathbf{u}$ and $\mathbf{v}$ for $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ to be orthogonal. **[4]**

Question 4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map corresponding to the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 4 & 2 \end{pmatrix},$$

and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map corresponding to the matrix

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) Calculate $f\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$. **[3]**

(b) Calculate the matrix product BA. What linear map (described in terms of f and g) does the matrix BA correspond to? **[4]**

(c) Calculate the inverse matrix B^{-1}. What linear map (described in terms of g) does the matrix B^{-1} correspond to? **[4]**

(d) Say why the sum $A + B$ is not defined. **[3]**

(e) Let C be a matrix such that the matrix product BAC corresponds to a linear map from $\mathbb{R}^2$ to itself. How many rows and columns must C have? **[3]**

(f) Let D be a matrix such that both the matrix product BAD and the matrix product DBA are meaningful expressions. How many rows and columns must D have? **[3]**

Question 5.

(a) Define precisely what it means for the non-zero vector $\mathbf{u} \in \mathbb{R}^3$ to be an **eigenvector** with **eigenvalue** λ for the 3×3 matrix M. [**6**]

(b) Define the **characteristic polynomial** of the matrix M. State (without proof) how the eigenvalues of M are determined by its characteristic polynomial. [**4**]

Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -3 \\ 2 & -2 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

(c) Verify that $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for A. What are the corresponding eigenvalues? [**5**]

(d) Determine $A^{100}\mathbf{w}$ and $A^{101}\mathbf{w}$, showing all steps of your argument carefully. [**5**]

End of Paper.