ECOM073: Topics in Financial Econometrics

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Exercise 6.

Problem 6.1. Consider AR(1) model

$$X_t = \phi X_{t-1} + \varepsilon_t$$

where ε_t is a white noise sequence with zero mean and variance σ_ε^2.

(1) Find the 1-step ahead forecast of X_{t+1}. Find the variance of the 1-step ahead forecast.

(2) Find the 2-step ahead forecast of X_{t+2}. Find the variance of the 1-step ahead forecast.

(3) Find the k-step ahead forecast of X_{t+k}. Find the variance of the k-step ahead forecast. Comment on properties of this forecast when k increases.

Solution 1. The k-step ahead forecast is defined by the formula

$$\hat{X}_t(k) = E[X_{t+k}|F_t] = [X_{t+k}],$$

using notation $[X_{t+k}]$.

(1) To compute $\hat{X}_t(1)$ first we write

$$X_{t+1} = \phi X_t + \varepsilon_{t+1}.$$

Then, using forecasting rules about $[X_s]$ and $[\varepsilon_s]$,

$$\begin{aligned}
\hat{X}_t(1) &= E[X_{t+1}|F_t] = [X_{t+1}] \\
&= [\phi X_t + \varepsilon_{t+1}] \\
&= \phi[X_t] + [\varepsilon_{t+1}] \\
&= \phi X_t.
\end{aligned}$$

Here we used the facts

$$[X_t] = X_t$$

which hold because X_t is known when we know F_t, and

$$[\varepsilon_{t+1}] = 0$$

which holds because ε_{t+1} is independent of the 'history' F_t.

Therefore, the 1-step ahead forecast is

$$\hat{X}_t(1) = \phi X_t.$$

The error of 1-step ahead forecast is

$$\begin{aligned} e_t(1) &= X_{t+1} - \hat{X}_t(1) \\ &= \phi X_t + \varepsilon_{t+1} - \{\phi X_t\} \\ &= \varepsilon_{t+1}. \end{aligned}$$

The variance of the error is

$$Var(e_t(1)) = Var(\varepsilon_{t+1}) = \sigma_\varepsilon^2.$$

(2) To compute 2-step ahead forecast, write

$$X_{t+2} = \phi X_{t+1} + \varepsilon_{t+2}.$$

Then

$$\begin{aligned} \hat{X}_t(2) &= [X_{t+2}] = [\phi X_{t+1} + \varepsilon_{t+2}] \\ &= \phi[X_{t+1}] \\ &= \phi \hat{X}_t(1) = \phi^2 X_t. \end{aligned}$$

The error of 2-step ahead forecast is

$$\begin{aligned} e_t(2) &= X_{t+2} - \hat{X}_t(2) \\ &= \phi X_{t+1} + \varepsilon_{t+2} - \phi \hat{X}_t(1) \\ &= \phi(X_{t+1} - \hat{X}_t(1)) + \varepsilon_{t+2} = \phi\varepsilon_{t+1} + \varepsilon_{t+2}. \end{aligned}$$

The variance of the error is

$$\begin{aligned} Var(e_t(2)) &= Var(\phi\varepsilon_{t+1} + \varepsilon_{t+2}) = Var(\phi\varepsilon_{t+1}) + Var(\varepsilon_{t+2}) \\ &= \phi^2\sigma_\varepsilon^2 + \sigma_\varepsilon^2 = (\phi^2 + 1)\sigma_\varepsilon^2. \end{aligned}$$

We conclude that

$$Var(e_t(2)) > Var(e_t(1)).$$

(2) To compute k-step ahead forecast, write

$$X_{t+k} = \phi X_{t+k-1} + \varepsilon_{t+k}.$$

Then

$$\begin{aligned} \hat{X}_t(k) &= [X_{t+k}] = [\phi X_{t+k-1} + \varepsilon_{t+k}] \\ &= \phi[X_{t+k-1}] \\ &= \phi\hat{X}_t(k-1) = \phi^2\hat{X}_t(k-2) = \phi^3\hat{X}_t(k-3) = \cdots = \phi^k\hat{X}_t(0) = \phi^k X_t. \end{aligned}$$

We see that

$$\hat{X}_t(k) = \phi^k X_t \to 0,$$

as k increases. This is the mean reversion property since the mean of X_t is $E[X_t] = 0$.

Problem 6.2. The researcher analyzed the sample $X_1, \cdots, X_t$ and found that it is not from a stationary time series. He checked the differenced series $z_t = X_t - X_{t-1}$ and found that fitting to it the AR(1) model gives uncorrelated residuals.

The fitted model was $z_t = 2 + 0.4z_{t-1} + \varepsilon_t$.

He/she is interested in forecasting X_{t+1} and X_{t+2}. Compute these forecasts

Solution. First we compute the 1-step and 2-step ahead forecasts of z_{t+1} and z_{t+2}.

To compute $\hat{z}_t(1)$, write $z_{t+1} = 2 + 0.4z_t + \varepsilon_{t+1}$. Then

$$\begin{aligned}\hat{z}_t(1) &= [2 + 0.4z_t + \varepsilon_{t+1}] \\ &= 2 + 0.4[z_t] + [\varepsilon_{t+1}] \\ &= 2 + 0.4z_t = 2 + 0.4(X_t - X_{t-1}).\end{aligned}$$

To compute $\hat{z}_t(2)$, write $z_{t+2} = 2 + 0.4z_{t+1} + \varepsilon_{t+2}$. Then

$$\begin{aligned}\hat{z}_t(2) &= [2 + 0.4z_{t+1} + \varepsilon_{t+2}] \\ &= 2 + 0.4[z_{t+1}] + [\varepsilon_{t+2}] \\ &= 2 + 0.4\hat{z}_t(1) \\ &= 2 + 0.4(2 + 0.4z_t) = 2.8 + 0.16z_t = 2.8 + 0.16(X_t - X_{t-1}).\end{aligned}$$

To compute $\hat{X}_t(1)$, we use equality $X_{t+1} = X_t + z_{t+1}$.

Then

$$\begin{aligned}\hat{X}_t(1) &= [X_{t+1}] = [X_t + z_{t+1}] = [X_t] + [z_{t+1}] \\ &= X_t + \hat{z}_t(1) \\ &= X_t + \{2 + 0.4(X_t - X_{t-1})\} \\ &= 2 + 1.4X_t - 0.4X_{t-1}.\end{aligned}$$

Then

$$\begin{aligned}\hat{X}_t(2) &= [X_{t+2}] = [X_{t+1} + z_{t+2}] \\ &= \hat{X}_t(1) + \hat{z}_t(2) = \{2 + 1.4X_t - 0.4X_{t-1}\} + \{2.8 + 0.16(X_t - X_{t-1})\} \\ &= 4.8 + 1.56X_t - 0.56X_{t-1}.\end{aligned}$$

Problem 6.3. Consider MA(2) model

$$X_t = c_0 + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$$

where ε_t is a white noise sequence with zero mean and variance σ_ε^2.

(1) Find the 1-step, 2-step and 3-step-ahead forecasts.

(2) Show that the forecast reverts to the mean at the step $k = 3$.

(3) Find the forecast errors.

Solution . **(1)** To compute $\hat{X}_t(1)$, use the rules we used in previous exercises. then

$$\begin{aligned}
\hat{X}_t(1) &= [c_0 + \varepsilon_{t+1} + \theta_1\varepsilon_t + \theta_2\varepsilon_{t-1}] \\
&= c_0 + [\varepsilon_{t+1}] + \theta_1[\varepsilon_t] + \theta_2[\varepsilon_{t-1}] \\
&= c_0 + \theta_1\varepsilon_t + \theta_2\varepsilon_{t-1}; \\
\hat{X}_t(2) &= [c_0 + \varepsilon_{t+2} + \theta_1\varepsilon_{t+1} + \theta_2\varepsilon_t] \\
&= c_0 + [\varepsilon_{t+2}] + \theta_1[\varepsilon_{t+1}] + \theta_2[\varepsilon_t] \\
&= c_0 + \theta_2\varepsilon_t; \\
\hat{X}_t(3) &= [c_0 + \varepsilon_{t+3} + \theta_1\varepsilon_{t+2} + \theta_2\varepsilon_{t+1}] \\
&= c_0 + [\varepsilon_{t+3}] + \theta_1[\varepsilon_{t+2}] + \theta_2[\varepsilon_{t+1}] \\
&= c_0.
\end{aligned}$$

Notice, that

$$EX_t = E[c_0 + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}] = c_0 + E[\varepsilon_t] + \theta_1 E[\varepsilon_{t-1}] + \theta_2 E[\varepsilon_{t-2}] = c_0.$$

Therefore, forecast $\hat{X}_t(3) = c_0$ reverted to the mean at the step $k = 3$.

(2) Next we compute forecast errors.

$$\begin{aligned}
e_t(1) = X_{t+1} - \hat{X}_t(1) &= \{c_0 + \varepsilon_{t+1} + \theta_1\varepsilon_t + \theta_2\varepsilon_{t-1}\} - \{c_0 + \theta_1\varepsilon_t + \theta_2\varepsilon_{t-1}\} = \varepsilon_{t+1}; \\
Var(e_t(1)) &= Var(\varepsilon_{t+1}) = \sigma_\varepsilon^2; \\
e_t(2) = X_{t+2} - \hat{X}_t(2) &= \{c_0 + \varepsilon_{t+2} + \theta_1\varepsilon_{t+1} + \theta_2\varepsilon_t\} - \{c_0 + \theta_2\varepsilon_t\} = \varepsilon_{t+1} + \theta_1\varepsilon_{t+1}; \\
Var(e_t(2)) &= Var(\varepsilon_{t+1} + \theta_1\varepsilon_{t+1}) = \sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2; \\
e_t(3) = X_{t+3} - \hat{X}_t(3) &= \{c_0 + \varepsilon_{t+3} + \theta_1\varepsilon_{t+2} + \theta_2\varepsilon_{t+1}\} - \{c_0\} = \varepsilon_{t+3} + \theta_1\varepsilon_{t+2} + \theta_2\varepsilon_{t+1}; \\
Var(e_t(3)) &= Var(\varepsilon_{t+3} + \theta_1\varepsilon_{t+2} + \theta_2\varepsilon_{t+1}) = \sigma_\varepsilon^2 + \theta_1^2\sigma_\varepsilon^2 + \theta_2^2\sigma_\varepsilon^2 \\
&= \sigma_\varepsilon^2(1 + \theta_1^2 + \theta_2^2).
\end{aligned}$$

Problem 6.4. Consider AR(2) model

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

where ε_t is a white noise sequence with zero mean and variance 1.

(1) Find the 1-step ahead forecast of X_{t+1}

(2) Find the 2-step ahead forecast of X_{t+2}

(3) Find the variance of the 1-step ahead forecast and the variance of 2-step ahead forecast. Compare these variances.

Solution 1. The k-step ahead forecast is defined by the formula

$$\hat{X}_t(k) = E[X_{t+k}|F_t] = [X_{t+k}],$$

using notation $[X_{t+k}]$.

(1) To compute $\hat{X}_t(1)$ first we write

$$X_{t+1} = \phi_0 + \phi_1 X_t + \phi_2 X_{t-1} + \varepsilon_{t+1}.$$

Then, using forecasting rules about $[X_s]$ and $[\varepsilon_s]$,

$$\begin{aligned} \hat{X}_t(1) &= E[X_{t+1}|F_t] = [X_{t+1}] \\ &= [\phi_0 + \phi_1 X_t + \phi_2 X_{t-1} + \varepsilon_{t+1}] \\ &= \phi_0 + \phi_1 [X_t] + \phi_2 [X_{t-1}] + [\varepsilon_{t+1}] \\ &= \phi_0 + \phi_1 X_t + \phi_2 X_{t-1}. \end{aligned}$$

Here we used the facts

$$[X_t] = X_t, \quad [X_{t-1}] = X_{t-1}$$

which hold because X_t and X_{t-1} are known when we know F_t, and

$$[\varepsilon_{t+1}] = 0$$

which is valid because ε_{t+1} is independent of the 'history' F_t.

Therefore, the 1-step ahead forecast is

$$\hat{X}_t(1) = \phi_0 + \phi_1 X_t + \phi_2 X_{t-1}.$$

(2) To compute 2-step ahead forecast, write

$$X_{t+2} = \phi_0 + \phi_1 X_{t+1} + \phi_2 X_t + \varepsilon_{t+2}.$$

Then

$$\begin{aligned}
\hat{X}_t(2) &= [X_{t+2}] = [\phi_0 + \phi_1 X_{t+1} + \phi_2 X_t + \varepsilon_{t+2}] \\
&= \phi_0 + \phi_1 [X_{t+1}] + \phi_2 [X_t] \\
&= \phi_0 + \phi_1 \hat{X}_t(1) + \phi_2 X_t.
\end{aligned}$$

We found that

$$\hat{X}_t(1) = \phi_0 + \phi_1 X_t + \phi_2 X_{t-1}.$$

So,

$$\begin{aligned}
\hat{X}_t(2) &= \phi_0 + \phi_1\{\phi_0 + \phi_1 X_t + \phi_2 X_{t-1}\} + \phi_2 X_t \\
&= \phi_0 + \phi_1\phi_0 + (\phi_1^2 + \phi_2) X_t + \phi_1\phi_2 X_{t-1}.
\end{aligned}$$

(3) The error of 1-step ahead forecast is

$$\begin{aligned}
e_t(1) &= X_{t+1} - \hat{X}_t(1) \\
&= \{\phi_0 + \phi_1 X_t + \phi_2 X_{t-1} + \varepsilon_{t+1}\} - \{\phi_0 + \phi_1 X_t + \phi_2 X_{t-1}\} \\
&= \varepsilon_{t+1}.
\end{aligned}$$

The variance of the error is

$$Var(e_t(1)) = Var(\varepsilon_{t+1}) = \sigma_\varepsilon^2 = 1.$$

The error of 2-step ahead forecast is

$$\begin{aligned}
e_t(2) &= X_{t+2} - \hat{X}_t(2) \\
&= \{\phi_0 + \phi_1 X_{t+1} + \phi_2 X_t + \varepsilon_{t+2}\} - \{\phi_0 + \phi_1 \hat{X}_t(1) + \phi_2 X_t\} \\
&= \phi_1(X_{t+1} - \hat{X}_t(1)) + \varepsilon_{t+2} = \phi_1 \varepsilon_{t+1} + \varepsilon_{t+2}.
\end{aligned}$$

The variance of the error is

$$\begin{aligned}
Var(e_t(2)) &= Var(\phi_1 \varepsilon_{t+1} + \varepsilon_{t+2}) = Var(\phi_1 \varepsilon_{t+1}) + Var(\varepsilon_{t+2}) \\
&= \phi_1^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2 = \phi_1^2 + 1.
\end{aligned}$$

We conclude that

$$Var(e_t(2)) > Var(e_t(1)).$$