ECOM073: Topics in Financial Econometrics

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Exercise 5.

Problem 5.1. Consider a stationary AR(1) process

$$X_t = \phi X_{t-1} + \varepsilon_t,$$

where the process ε_t is white noise process with zero mean and variance $E\varepsilon_t^2 = \sigma_\varepsilon^2$, and $|\phi| < 1$

Prove the following

(i) $EX_t = 0$.
(ii) $Var(X_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2}$.
(iii) Show that autocovariance function is

$$\gamma_k = \frac{\sigma_\varepsilon^2}{1-\phi^2}\phi^k, \qquad k = 0, 1, 2, \cdots.$$

Show that autocorrelation function

$$\rho_k = \phi^k, \qquad k = 0, 1, 2, \cdots.$$

Solution:
(i) We take expectation of both side of AR(1) equation:

$$\begin{aligned} E[X_t] &= E[\phi X_{t-1} + \varepsilon_t] \\ &= E[\phi X_{t-1}] + E[\varepsilon_t] \\ &= \phi E[X_{t-1}] \end{aligned}$$

since $E[\varepsilon_t] = 0$. Since for $|\phi| < 1$, X_t is a stationary process, then $E[X_t] = E[X_{t-1}] = \mu$ does not depend on time t. Therefore

$$\mu = \phi\mu, \quad or \quad \mu = \frac{0}{1-\phi} = 0.$$

(ii) We showed that $EX_t = 0$. So, by definition

$$\begin{aligned} Var(X_t) &= E(X_t - E[X_t])^2 = EX_t^2 = E(\phi X_{t-1} + \varepsilon_t)^2 \\ &= E(\phi^2 X_{t-1}^2 + 2\phi X_{t-1}\varepsilon_t + \varepsilon_t^2) \\ &= \phi^2 EX_{t-1}^2 + 2\phi E[X_{t-1}\varepsilon_t] + E[\varepsilon_t^2]. \end{aligned}$$

Since time series X_t is stationary, its variance remains constant: $Var(X_t) = EX_t^2 = EX_{t-1}^2 = \sigma_Y^2$. Moreover, future is not correlated with the past, so $E[X_{t-1}\varepsilon_t] = 0$. Thus we obtain

$$\sigma_Y^2 = \phi^2\sigma_Y^2 + \sigma_\varepsilon^2, \quad or \quad \sigma_Y^2 = \frac{\sigma_\varepsilon^2}{1-\phi^2}.$$

(iii). Since $EX_t = 0$, then for $k \geq 1$,

$$\begin{aligned} \gamma_k &= Cov(X_t, X_{t-k}) = E[(X_t - EX_t)(X_{t-k} - EX_{t-k})] \\ &= E[X_t X_{t-k}]. \end{aligned}$$

Since $X_t = \phi X_{t-1} + \varepsilon_t$, then

$$\begin{aligned} \gamma_k &= E[X_t X_{t-k}] = E[(\phi X_{t-1} + \varepsilon_t)X_{t-k}] \\ &= \phi E[X_{t-1}X_{t-k}] + E[\varepsilon_t X_{t-k}] \\ &= \phi E[X_{t-1}X_{t-k}] \end{aligned}$$

because white noise ε_t is uncorrelated with the past and therefore $E[\varepsilon_t X_{t-k}] = 0$. Because of stationarity,

$$\gamma_k = Cov(X_t, X_{t-k}) = E[X_t X_{t-k}], \quad \gamma_{k-1} = E[X_{t-1}X_{t-k}]$$

and we obtain

$$\gamma_k = \phi\gamma_{k-1}, \quad \text{for all} \quad k \geq 0.$$

From here, we deduce that

$$\gamma_k = \phi^2\gamma_{k-2} = \ldots = \phi^k\gamma_0, \quad k \geq 0.$$

By definition $\rho_k = \gamma_k/\gamma_0$. Then

$$\rho_0 = 1$$

$$\rho_1 = \phi$$
$$\rho_2 = \phi^2$$
$$\cdots\cdots$$
$$\rho_k = \phi^k.$$

Note that differently from autocovariance γ_k, autocorrelation ρ_k does not depend on the variance of the white noise ε_t.

Problem 5.2. A national bank started accepting electronic checks over the Internet in January 2006. Prior to that data, only paper checks were accepted. A local branch collected the data on weekly number of paper checks processed at the branch from January 2004 to January 2008. Consider only the first two years of that data set, and fit an appropriate ARMA model.

Solution: ACF and PACF analysis shows that we can fit either AR(1) model or MA(3) model. Here the rational selection would be AR(1) (simplest model).

Note: E-views provide no option for automatic selection of the order p, q for fitting ARMA (p,q) model. If we wont to use AIC model selection criterion or BIC(Schwarz) criterion, we have to do that manually: fit different models and check which minimizes AIC or BIC.

For example, fitting AR(1), MA(3) and ARMA(1,1) models to this data we obtain the following values of AIC and BIC criterions (see outputs below):

	AIC	BIC (Schwarz)
$AR(1)$	8.7383	8.789
$MA(3)$	8.771	8.873
$ARMA(1,1)$	8.74	8.8824

Date: 02/14/12 Time: 17:58
Sample: 1 105
Included observations: 105

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.613	0.613	40.577	0.000
		2	0.410	0.056	58.954	0.000
		3	0.274	0.003	67.201	0.000
		4	0.119	-0.102	68.765	0.000
		5	0.064	0.026	69.220	0.000
		6	-0.025	-0.086	69.290	0.000

Conclusion on model selection:

- AIC criterion suggest the following order: AR(1) is fitting best, than MA(3), then ARMA(1,1).
- BIC criterion suggests the following order: AR(1) is fitting best, then ARMA(1,1), than MA(3).

So for fitting to the data and forecasting we may go for an AR(1) model.

For illustration, we also fit MA(3) and ARMA(1,1) models.

Fitting AR(1) model and using it for forecasting.

The below outputs of estimation of AR(1) model, residual check and forecasting show:

- The AR(1) model is

$$X_t = 498.18 + 0.61X_{t-1} + \varepsilon_t, \qquad \sigma_\varepsilon = 18.92,$$

- AR(1) coefficient $\phi = 0.615$ is significant
- residuals are not correlated, so the model is fitting well
- Forecasting graph shows the values out of the sample forecasts, i.e. 1,2, 3 step ahead forecasts.

  Observe the following pattern: when the step k increases, the forecast reverts to the mean which is about 500, as it should be according the theory.

  The graph also shows 95% confidence band for the forecasted values.

Dependent Variable: SERIES01
Method: Least Squares
Date: 02/14/12 Time: 18:35
Sample (adjusted): 2 105
Included observations: 104 after adjustments
Convergence achieved after 3 iterations

AR(1) PROCESS

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	498.1845	4.823037	103.2927	0.0000
AR(1)	0.615130	0.077790	7.907574	0.0000

R-squared	0.380051	Mean dependent var	498.0769
Adjusted R-squared	0.373973	S.D. dependent var	23.92451
S.E. of regression	18.92952	Akaike info criterion	8.738366
Sum squared resid	36549.32	Schwarz criterion	8.789219
Log likelihood	-452.3950	Hannan-Quinn criter.	8.758968
F-statistic	62.52972	Durbin-Watson stat	2.063759
Prob(F-statistic)	0.000000		

Inverted AR Roots	.62

Residual diagnostic

EViews - [Equation: UNTITLED Workfile: ARMA_01::Arma_01\]

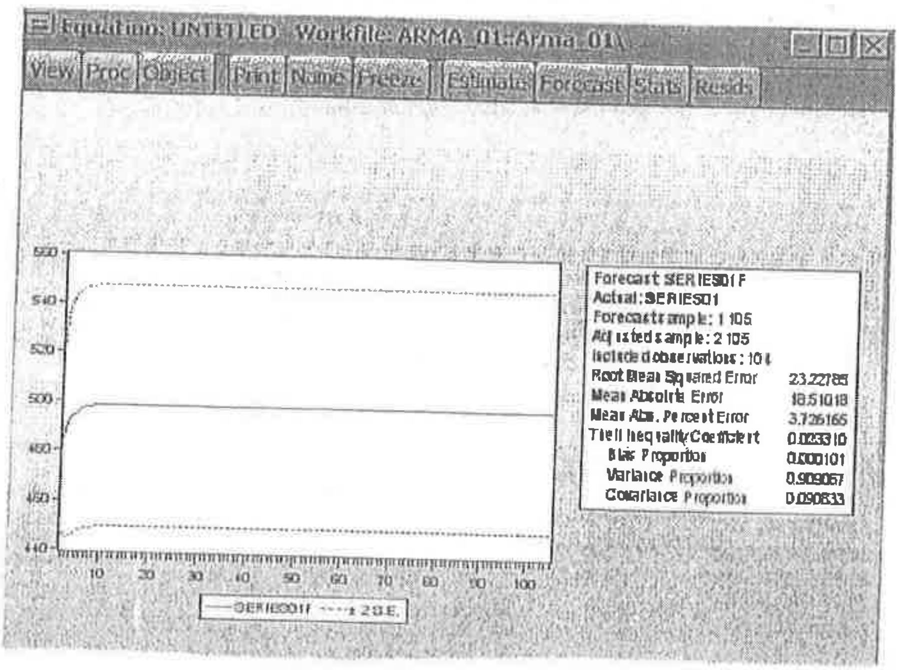
File Edit Object View Proc Quick Options Add-ins Window Help

View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Date: 02/14/12 Time: 19:02
Sample: 2 105
Included observations: 104
Q-statistic probabilities adjusted for 1 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.055	-0.055	0.3196	
		2	0.023	0.020	0.3781	0.539
		3	0.066	0.069	0.8581	0.651
		4	-0.077	-0.071	1.5087	0.680
		5	0.054	0.044	1.8320	0.767
		6	-0.019	-0.015	1.8725	0.866

Forescast



Fitting MA(3) model and using it for forecasting.

The below outputs of estimation of MA(3) model, residual check and forecasting show:

- The MA(3) model is

$$X_t = 497.7 + 0.589\varepsilon_{t-1} + 0.3667\varepsilon_{t-2} + 0.2643\varepsilon_{t-3} + \varepsilon_t, \qquad \sigma_\varepsilon = 19.07,$$

- All MA(3) coefficients are significant
- residuals are not correlated, so the model is fitting well
- Forecasting graph shows that forecast reverts to the mean ~ 500 after 3 steps, as it should be according the theory.

  That means, using MA(3) model for forecasting, forecasts with step $k = 4, 5, \ldots.$ ahead will be equal to the (sample) mean.

  The graph also shows 95% confidence band for the forecasted values.

FITTING A MA(3) MODEL

EViews - [Equation: UNTITLED Workfile: ARMA_01::Arma_01\]

File Edit Object View Proc Quick Options Add-ins Window Help

View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Dependent Variable: SERIES01
Method: Least Squares
Date: 02/14/12 Time: 18:43
Sample: 1 105
Included observations: 105
Convergence achieved after 8 iterations
MA Backcast: -2 0

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	497.7035	4.084209	121.8605	0.0000
MA(1)	0.589871	0.097044	6.078391	0.0000
MA(2)	0.366740	0.106790	3.434226	0.0009
MA(3)	0.246367	0.097697	2.521743	0.0132

R-squared	0.381726	Mean dependent var	497.8667
Adjusted R-squared	0.363362	S.D. dependent var	23.90650
S.E. of regression	19.07490	Akaike info criterion	8.771974
Sum squared resid	36749.04	Schwarz criterion	8.873078
Log likelihood	-456.5287	Hannan-Quinn criter.	8.812944
F-statistic	20.78602	Durbin-Watson stat	1.968758
Prob(F-statistic)	0.000000		

Inverted MA Roots	.02-.63i	.02+.63i	-.63

Residual diagnostic MA(3)

EViews - [Equation: UNTITLED Workfile: ARMA_01::Arma_01\]

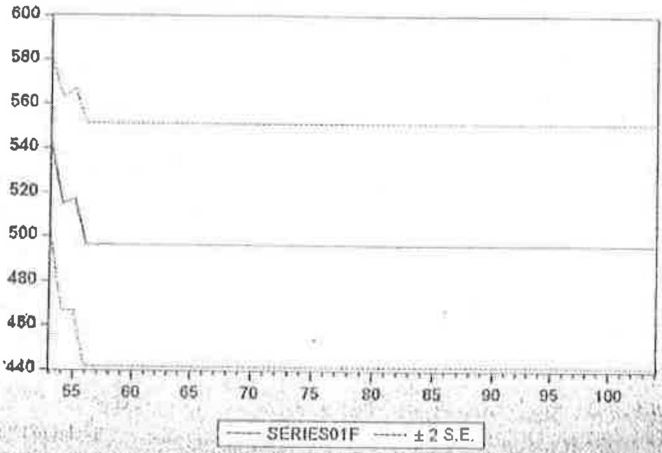
File Edit Object View Proc Quick Options Add-ins Window Help

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Date: 02/14/12 Time: 18:46
Sample: 1 105
Included observations: 105
Q-statistic probabilities adjusted for 3 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.007	0.007	0.0058	
		2	0.024	0.024	0.0685	
		3	0.051	0.050	0.3517	
		4	0.071	0.070	0.9165	0.338
		5	0.054	0.052	1.2473	0.536
		6	-0.024	-0.030	1.3126	0.726

Forecasting MA(3)
Dynamic



Forecast: SERIES01F	
Actual: SERIES01	
Forecast sample: 53 104	
Included observations: 52	
Root Mean Squared Error	21.33963
Mean Absolute Error	15.94591
Mean Abs. Percent Error	3.181944
Theil Inequality Coefficient	0.021347
Bias Proportion	0.019435
Variance Proportion	0.509666
Covariance Proportion	0.470898

Fitting ARMA(1,1) model and using it for forecasting.

The below outputs of estimation of ARMA(1,1) model, residual check and forecasting show:

- The AR(1) model is

$$X_t = 498.58 + 0.71X_{t-1} + \varepsilon_t - 0.017\varepsilon_{t-1}, \qquad \sigma_\varepsilon = 18.93,$$

- AR(1) coefficient $\phi = 0.71$ is significant, the moving average coefficient $\theta = -0.1722$ is not significant. That indicates we should use AR(1) model instead of ARMA(1,1). It tells us, we are overfitting.
- residuals are not correlated, so the model is fitting well.
- Forecasting graph shows the values out of the sample forecasts, i.e. 1,2, 3 step ahead forecasts.

  When the step k increases, the forecast reverts to the mean which is about 500, as it should be according the theory.

  The graph also shows 95% confidence band for the forecasted values.

EViews - [Equation: UNTITLED Workfile: ARMA_01::Arma_01\]
File Edit Object View Proc Quick Options Add-ins Window Help
View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: SERIES01
Method: Least Squares
Date: 02/14/12 Time: 18:21
Sample (adjusted): 2 105
Included observations: 104 after adjustments
Convergence achieved after 6 iterations
MA Backcast: 1

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	498.5841	5.382869	92.62423	0.0000
AR(1)	0.713541	0.108178	6.596017	0.0000
MA(1)	-0.172267	0.154165	-1.117422	0.2665

R-squared	0.386071	Mean dependent var	498.0769
Adjusted R-squared	0.373914	S.D. dependent var	23.92451
S.E. of regression	18.93041	Akaike info criterion	8.747838
Sum squared resid	36194.39	Schwarz criterion	8.824119
Log likelihood	-451.8876	Hannan-Quinn criter.	8.778741
F-statistic	31.75713	Durbin-Watson stat	1.927610
Prob(F-statistic)	0.000000		

Inverted AR Roots	.71
Inverted MA Roots	.17

RESIDUAL DIAGNOSTIC
ARMA(1,1)

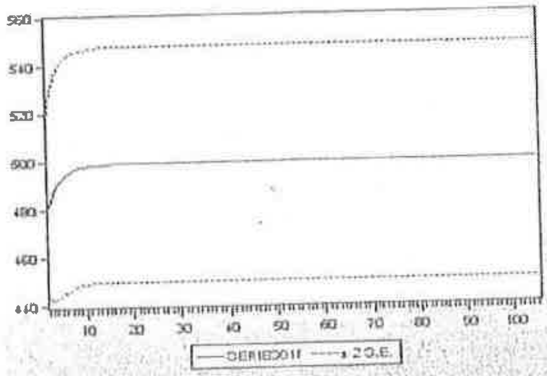
EViews - [Equation: UNTITLED Workfile: ARMA_01::Arma_01\]
File Edit Object View Proc Quick Options Add-ins Window Help
View Proc Object Print Name Freeze Estimate Forecast Stats Resids

Date: 02/14/12 Time: 21:40
Sample: 2 105
Included observations: 104
Q-statistic probabilities adjusted for 2 ARMA term(s)

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.017	0.017	0.0306	
		2	-0.022	-0.022	0.0830	
		3	0.022	0.023	0.1376	0.711
		4	-0.090	-0.091	1.0254	0.599
		5	0.033	0.038	1.1479	0.766
		6	-0.032	-0.039	1.2643	0.867

Using the whole sample to forecast ARMA(1,1)

Equation: UNTITLED Workfile: ARMA_01::Arma_01\
View Proc Object Print Name Freeze Estimate Forecast Stats Resids



Forecast: SERIES01F	
Actual: SERIES01	
Forecast sample: 1 105	
Adjusted sample: 2 105	
Included observations: 104	
Root Mean Squared Error	22.93561
Mean Absolute Error	16.29323
Mean Abs. Percent Error	3.682026
Theil Inequality Coefficient	0.023014
Bias Proportion	0.000009
Variance Proportion	0.876390
Covariance Proportion	0.123601