6.5 Bijections and cardinality

Suppose A and B are finite sets, and $f: A \to B$ is a function. If f is injective, what can we say about $|A|$ and $|B|$?

Theorem 6.3: Suppose A and B are finite sets, and $f: A \to B$ is a function.

(a) If f is injective, then $|A| \leq |B|$.

(b) If f is surjective, then $|A| \geq |B|$.

(c) If f is bijective, then $|A| = |B|$.

Pf: (a) Write the elements of A as $a_1, a_2, \dots, a_m$, where $m = |A|$. Because f is injective, $f(a_1), f(a_2), \dots, f(a_m)$ are all different. So B has at least m different elements, so $|B| \geq m$.

(b) Write the elements of B as $b_1, b_2, \dots, b_n$, where $n = |B|$. Because f is surjective, there exist elements $a_1, a_2, \dots, a_n \in A$ such that

$$f(a_1) = b_1, \ f(a_2) = b_2, \dots, f(a_n) = b_n.$$

The elements $a_1, a_2, \dots, a_n$ are all different (because they map to different elements of B), so A has at least n different elements, so $|A| \geq n$.

(c) f is injective, so $|A| \leq |B|$.
f is surjective, so $|A| \geq |B|$.
So $|A| = |B|$. $\square$

Remarks: (1) A stronger statement is true: if $|A| = |B|$, then f is injective iff it is surjective.

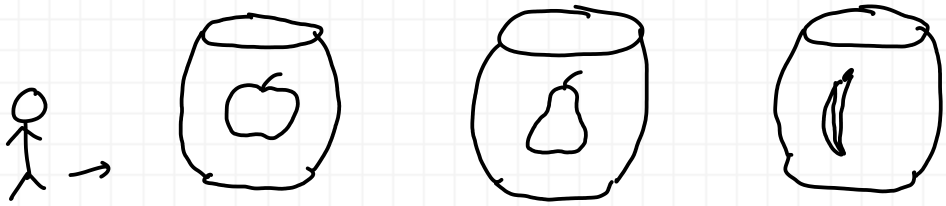
(2) Bijections are used to define cardinality for infinite sets: two sets have the same cardinality iff there is a bijection between them.

Part (c) of the theorem is often used in counting problems: we might have a finite set A and we want to know $|A|$, but the elements of A are hard to count. But we may be able to find a bijection $f: A \to B$ where B is a set such that $|B|$ is easy to find.

e.g. Suppose we have a barrel of apples
a barrel of pears
a barrel of bananas

and we want to choose 10 pieces of fruit (e.g. 6 apples, 3 pears, 1 banana). How many ways can we do this?

Imagine the barrels coming in order:



Making a choice consists of 12 step: each step is either: Take from the current barrel, or
Move to the next barrel.

So we have a bijection from the set of ways of choosing our fruit to the set of sequences consisting of ten Ts and two Ms.

e.g. 3 apples, 5 pears, 2 bananas $\mapsto$ TTTMTTTTTMTT.

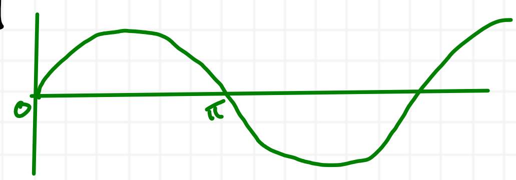
The number of sequences is $\binom{12}{2}$: once we know where the Ms go, the rest of the sequence is determined.

6.6 Images and inverse images of subsets

Def^n: Suppose $f: A \to B$ is a function, and $C \subseteq A$. The image of C under f is
$$f(C) = \{f(c) : c \in C\}.$$

e.g: • $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$.

$$f([0,\pi]) = [0,1]$$

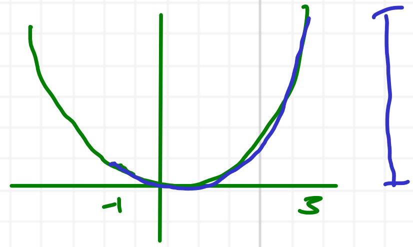


- $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

$$f(\{1,2,3\}) = \{1,4,9\}.$$

$$f([2,3]) = [4,9]$$

$$f([-1,3]) = [0,9]$$



- $f: \mathcal{P}(\{1,2,3,4\}) \to \{0,1,2,3,4\}$ given by $f(A) = |A|$.

$$f(\{X \subseteq \{1,2,3,4\} : 3 \in X\}) = \{1,2,3,4\}$$

$$f(\{X \subseteq \{1,2,3,4\} : 3 \in X, 4 \notin X\}) = \{1,2,3\}.$$

- For any $f: A \to B$,

$$f(A) = \text{range}(f)$$

$$f(\emptyset) = \emptyset.$$

Defn: Suppose $f: A \to B$ is a function, and $D \subseteq B$. The <u>inverse image</u> of D under f is

$$f^{-1}(D) = \{a \in A : f(a) \in D\}.$$