More examples:

- $f: \mathcal{P}(\{1,2,3,4\}) \to \{0,1,2,3,4\}$
  given by $f(A) = |A|$.

  f is surjective: given $b \in \{0,1,2,3,4\}$, let
  $A = \{1, 2, \ldots, b\}$. Then $f(A) = b$.

  f is not injective: $f(\{1,2\}) = 2 = f(\{2,3\})$.

- $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$
  defined by $f(a,b) = f(a+b, a-b)$.

  $f(0,0) = (0,0)$
  $f(1,2) = (3,-1)$
  $f(8,5) = (13,3)$
  $f(2,0) = (2,2)$
  $\vdots$

  f is not surjective: suppose $f(a,b) = (1,0)$.
  Then $a+b = 1$, $a-b = 0$
  $\Rightarrow a = b = 1/2$ ↯.

  f is injective: suppose $f(a,b) = f(c,d)$.
  Then $(a+b, a-b) = (c+d, c-d)$
  so $a+b = c+d$ and $a-b = c-d$.
  $\rightsquigarrow$ $a = c$ and $b = d$, so $(a,b) = (c,d)$.

- $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $f(a,b) = (a+b, a+2b)$.
  Then f is bijective (exercise).

6.3 Inverses

Defn: Suppose $f: A \to B$ is a function. An <u>inverse</u> of f is a function $g: B \to A$ such that

$$g(f(a)) = a \quad \text{for all } a \in A$$

<u>and</u>

$$f(g(b)) = b \quad \text{for all } b \in B.$$

f is <u>invertible</u> if it has an inverse.

e.g.
- $f: \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 3x - 4$.
  Then f has an inverse
  $g: \mathbb{Q} \to \mathbb{Q}$ given by $g(x) = \frac{x+4}{3}$.

Check: $g(f(x)) = g(3x-4) = \frac{3x-4+4}{3} = x.$

and $f(g(x)) = f(\frac{x+4}{3}) = 3\frac{x+4}{3} - 4 = x.$

- $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x-4$.
  f doesn't have an inverse: the inverse would have to be $x \mapsto \frac{x+4}{3}$. but this does not define a function from $\mathbb{Z}$ to $\mathbb{Z}$.

- $f: \mathbb{Q} \to \mathbb{Q}$ defined by $f(x) = 3-x$.

  f is its own inverse:
  $$f(f(x)) = 3-(3-x) = x.$$

- $f: \mathcal{P}(\{1,2,3,4\}) \to \{0,1,2,3,4\}$
  defined by $A \mapsto |A|$.
  f is not invertible: if g were an inverse for f,
  then $g(f(\{1,2\})) = \{1,2\}$ so $g(2) = \{1,2\}$
  but also $g(f(\{2,3\})) = \{2,3\}$ $g(2) = \{2,3\}$ $\lightning$.

- $f: \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = n-1$.
  f is not invertible: if g were an inverse, then

  $f(g(-1)) = -1$ but there is no n such that $f(n) = -1$. $\lightning$.

Lemma 6.1: Suppose $f: A \to B$ is invertible. Then f has a unique inverse.

General proof tips
(1) To prove that something is unique: suppose there are two of them, and show that they're the same.
(2) To show that two functions g and h are equal, they must have the same domain, the same codomain, and $g(x) = h(x)$ for every x.

Proof of Lemma 6.1: Suppose $g: B \to A$ and $h: B \to A$ are both inverses of f. We need to show that $g = h$.

g and h both have domain B and codomain A, so we need to show that $g(b) = h(b)$ for all $b \in B$.

We know $f(g(b)) = b$

Apply h to get $h(f(g(b))) = h(b)$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[l, bend right, "g"'] \arrow[l, bend left, "h"] \end{tikzcd}$$

But $h(f(a)) = a$ for all $a \in A$, so in particular
$$h(f(g(b))) = g(b).$$
so $h(b) = g(b)$. $\square$

In view of Lemma 6.1, we can talk about the inverse of an invertible function. We write this as f^{-1}.

e.g. • $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$.
Then $f^{-1} : [0, \infty) \to [0, \infty)$ is given by $f^{-1}(x) = \sqrt{x}$.

• $f : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ defined by $f(A) = \mathbb{Z} \setminus A$.
Then $f^{-1} = f$: given $A \subseteq \mathbb{Z}$,
$$\begin{aligned} f(f(A)) &= \mathbb{Z} \setminus (\mathbb{Z} \setminus A) \\ &= \mathbb{Z} \setminus \{n \in \mathbb{Z} : n \notin A\} \\ &= A. \end{aligned}$$

Theorem 6.2: Suppose $f : A \to B$ is a function. Then f is invertible iff it is bijective.

Pf: First suppose f is invertible. We need to show that f is both injective and surjective.

<u>f is injective:</u> suppose $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$.
Apply f^{-1} to get $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$
so $a_1 = a_2$.

<u>f is surjective:</u> given $b \in B$, let $a = f^{-1}(b)$.
Then $f(a) = f(f^{-1}(b)) = b$. So there exists $a \in A$ such that $f(a) = b$.

Conversely, suppose f is injective and surjective. We need to find an inverse of f.

Suppose $b \in B$. Since f is surjective, there is at least one $a \in A$ such that $f(a) = b$. Choose such an a, and let $g(b) = a$.

This defines a function $g : B \to A$. We need to check that g is an inverse of f.

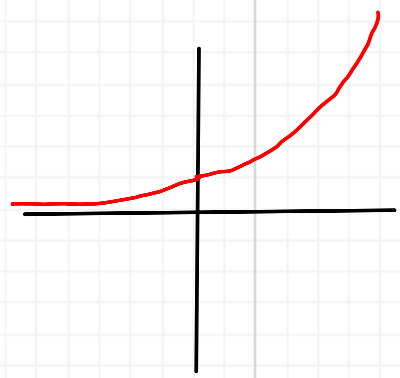
- if $b \in B$, then $f(g(b)) = b$, because $g(b)$ was chosen to satisfy this equation.
- if $a \in A$, let $b = f(a)$. Then $f(g(b)) = b$, so
$$f(g(f(a))) = f(a).$$
But f is injective, so if $f(a_1) = f(a_2)$ then $a_1 = a_2$. So in particular $g(f(a)) = a$.

So g is an inverse for f. $\square$

Examples

- Is there a bijection from $\mathbb{R}$ to $(0, \infty)$?

  Define $f : \mathbb{R} \to (0, \infty)$ by $f(x) = e^x$.



  Then f has an inverse
  $f^{-1} : (0, \infty) \to \mathbb{R}$
  $f^{-1}(x) = \log(x)$.

- Is there a bijection from $\mathbb{Z}$ to $\mathbb{N}$?

  Define $f : \mathbb{Z} \to \mathbb{N}$ by
$$f(n) = \begin{cases} \overset{\swarrow \text{odd}}{1 - 2n} & (n \leq 0) \\ \underset{\text{even}}{2n} & (n > 0). \end{cases}$$

$\vdots$
$f(3) = 6$
$f(2) = 4$
$f(1) = 2$
$f(0) = 1$
$f(-1) = 3$
$f(-2) = 5$
$f(-3) = 7$

  Then f has an inverse $f^{-1} : \mathbb{N} \to \mathbb{Z}$ given by
$$f^{-1}(n) = \begin{cases} \frac{1-n}{2} & (n \text{ odd}) \\ n/2 & (n \text{ even}). \end{cases}$$

$f^{-1}(1) = 0$
$f^{-1}(2) = 1$
$f^{-1}(3) = -1$
$f^{-1}(4) = 2$
$\vdots$

Let's check that f^{-1} really is an inverse:

if $n \in \mathbb{Z}$, then $f^{-1}(f(n)) = \begin{cases} f^{-1}(1-2n) & (n \le 0) \\ f^{-1}(2n) & (n > 0) \end{cases}$

$$= \begin{cases} n & (n \le 0) \\ n & (n > 0) \end{cases}$$

$$= n.$$

if $n \in \mathbb{N}$, then $f(f^{-1}(n)) = \begin{cases} f\left(\frac{1-n}{2}\right) & (n \text{ odd}) \\ f\left(\frac{n}{2}\right) & (n \text{ even}) \end{cases}$

$$= \begin{cases} n & (n \text{ odd}) \\ n & (n \text{ even}) \end{cases}$$

$$= n.$$

6.4 Restriction and composition of functions

Def^n: Suppose $f : A \to B$ is a function, and $D \subseteq A$. The restriction of f to D is the function $g : D \to B$ defined by $g(d) = f(d)$ for all $d \in D$. We write this function as $f|_D$.

eg
- $f : \{1, 2, 3, 4\} \to \{5, 6, 7\}$ defined by
  $f(1) = 6$, $f(2) = 7$, $f(3) = 5$, $f(4) = 6$.

  Let $D = \{1, 2, 3\}$. Then $f|_D$ is the function
  $g : \{1, 2, 3\} \to \{5, 6, 7\}$ given by
  $g(1) = 6$, $g(2) = 7$, $g(3) = 5$.

  (observe that f is not injective, but $f|_D$ is.)
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(\pi x)$.
  Then $f|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{R}$ is the constant function
  $f(n) = 0$ for all $n \in \mathbb{Z}$.

- $f : \{2, 3, 4, \ldots\} \to \mathbb{N}$ defined by
  $f(n) =$ the largest divisor of n apart from n itself.
  e.g. $f(15) = 5$, $f(20) = 10$, $f(17) = 1$, $f(91) = 13$.

  $f(2) = 1$, $f(4) = 2$, $f(6) = 3$, $f(8) = 4$, ..
  So if $D = \{2, 4, 6, 8, ..\}$, then $f|_D$ is given by $n \mapsto \frac{n}{2}$.

(Here f is surjective but not injective, $f|_D$ is bijective.)

Defn: Suppose A, B, C are sets, and $f: A \to B$ and $g: B \to C$ are functions. The composition of f and g is the function $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$.

Note that $g \circ f$ is only defined if the domain of g equals the codomain of f. We often just write gf instead of $g \circ f$.

e.g.

- $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$
  $g: \mathbb{N} \to \mathbb{N}$ defined by $g(n) = n+2$.

  Then $g \circ f: \mathbb{N} \to \mathbb{N}$ is given by $g \circ f(n) = 2n+2$.
  $f \circ g: \mathbb{N} \to \mathbb{N}$ is given by $f \circ g(n) = 2n+4$.

  (So even if $f \circ g$ and $g \circ f$ are both defined, they may be different).

- $f: \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^2$.
  $g: [0, \infty) \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$.

  Then $f \circ g: [0, \infty) \to [0, \infty)$ is given by $f \circ g(x) = (\sqrt{x})^2 = x$.
  But $g \circ f: \mathbb{R} \to \mathbb{R}$ is given by $g \circ f(x) = \sqrt{x^2} = |x|$.

- $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^3$.
  $g: \mathbb{N} \to \mathbb{N}$ defined by $g(n) = 3^n$.
  $g \circ f: \mathbb{N} \to \mathbb{N}$ is given by $g \circ f(n) = 3^{n^3}$
  $f \circ g: \mathbb{N} \to \mathbb{N}$ is given by $f \circ g(n) = (3^n)^3 = 3^{3n}$.

We can express the definition of inverses in terms of composition: given a set A, the identity function on A is the function $i: A \to A$ defined by $i(a) = a$.

Then: if $f: A \to B$ is a function, an inverse of f is a function $g: B \to A$ such that

$g \circ f$ is the identity function on A, and
$f \circ g$ is the identity function on B.