Non-examples: The following attempts <u>fail</u> to define a function:

- $f: \mathbb{Z} \to \mathbb{N}$ defined by $x \mapsto x^2$.
  (fails because $f(0) \notin \mathbb{N}$).

- $f: \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \frac{1}{x}$.
  ($f(0)$ is not defined)

- $f: \mathbb{N} \to \mathbb{R}$ defined by $x \mapsto \pm\sqrt{x}$.
  ($f(x)$ has two values when $x>0$).

- $f: \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x<0. \end{cases}$
  ($f(0)$ not defined)

- $f: \mathcal{P}(\mathbb{N}) \to \mathbb{Z}$ defined by $A \mapsto |A|$.
  ($f(\mathbb{N}) \notin \mathbb{Z}$).

Def^n: Suppose $f: A \to B$ is a function. The <u>range</u> of f is the set of all values of f:

$$\text{range}(f) = \{f(a) : a \in A\}.$$

Remark: The <u>codomain</u> of f is the set of "allowed" values of f. The range is the set of values it actually takes.

In some subjects (especially in algebra), the range is often called the <u>image</u> of f.

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e.g.

- $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
  $\text{range}(f) = \{x \in \mathbb{R} : x \geq 0\}$
- $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$.
  $\text{range}(f) = \{n \in \mathbb{N} : 2 \mid n\}$
  (i.e. the set of even natural numbers).
- $f: \{1,2,3\} \to \{4,5,6\}$ defined by
  $f(1) = 5$, $f(2) = 6$, $f(3) = 6$.
  $\text{range}(f) = \{5, 6\}$.

- $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x)$.

$$\text{range}(f) = [-1, 1] = \{x \in \mathbb{R} : -1 \le x \le 1\}.$$

- $f: \mathcal{P}(\{1,2,3,4\}) \to \mathbb{Z}$ defined by $A \mapsto |A|$.

$$\text{range}(f) = \{0, 1, 2, 3, 4\}.$$

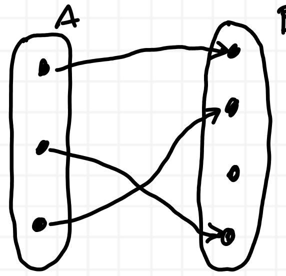
6.2 Injective, surjective, bijective

Def^n: Suppose $f: A \to B$ is a function. f is:

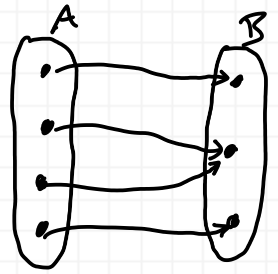
- <u>injective</u> if $f(a_1) \neq f(a_2)$ whenever $a_1, a_2 \in A$ and $a_1 \neq a_2$.
- <u>surjective</u> if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.
- <u>bijective</u> if it is both injective and surjective.

A bijective function is called a <u>bijection</u>.

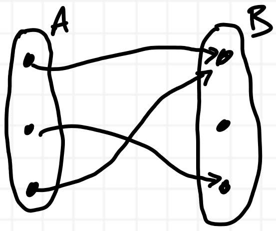
We can visualise these properties as follows



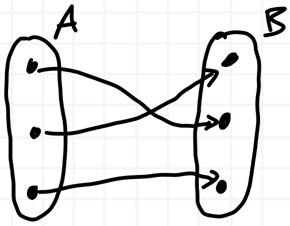
injective but not surjective



surjective but not injective



neither injective nor surjective



both injective and surjective

Tips for proving that a function is (not) injective/surjective:

- to prove f is not surjective: give a specific $b \in B$ such that there is no $a \in A$ for which $f(a) = b$.
- to prove f is not injective: give specific $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$.
- to prove f is surjective: explain how to find, for general $b \in B$, an element $a \in A$ such that $f(a) = b$.
- to prove f is injective: use the contrapositive: show that if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

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e.g.

- $f: \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$.

  not injective: $f(1) = f(-1)$.

  not surjective: there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

- $f: \{1, 2, 3\} \to \{\text{red}, \text{blue}\}$ defined by

  $f(1) = \text{red}$, $f(2) = \text{blue}$, $f(3) = \text{blue}$.

  not injective: $f(2) = f(3)$

  surjective: $\text{red} = f(1)$

  $\text{blue} = f(2)$

- $f: \mathbb{N} \to \mathbb{N}$ defined by $x \mapsto x^2$.

  injective: to see this, prove the contrapositive.

  Suppose $a_1, a_2 \in \mathbb{N}$ and $f(a_1) = f(a_2)$

  Then $a_1^2 = a_2^2$

  so $a_1^2 - a_2^2 = 0$

  so $(a_1 + a_2)(a_1 - a_2) = 0$.

  So either $a_1 = a_2$ or $a_1 = -a_2$.

  But a_1, a_2 are positive, so $a_1 \neq -a_2$.

  So $a_1 = a_2$.

  not surjective: there is no $a \in \mathbb{N}$ such that $f(a) = 3$.

- $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$.

  not injective: $f(0) = f(\pi)$.

  not surjective: there is no $x \in \mathbb{R}$ such that $\sin(x) = 2$.