Now suppose we have three sets A, B, C.
Then we can define the Cartesian product $A \times B \times C$ to be the set of all <u>ordered triples</u>

$$A \times B \times C = \{(a,b,c) : a \in A,\ b \in B,\ c \in C\}.$$

We can extend this to more than three sets.
Often the sets will be the same. We write

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ copies of } A}.$$

e.g. $\mathbb{R}^3 = \{(x,y,z) : x, y, z \in \mathbb{R}\}$ is often identified with 3-dimensional space via Cartesian coordinates.

5.5 Cardinality

Defn: If A is a finite set, the <u>cardinality</u> of A is the number of elements in A.
We write $|A|$ for the cardinality of A.

e.g.
- $|\{1,3,5\}| = 3$.
- $|\emptyset| = 0$.
- if $n \in \mathbb{N}$, then $|\{n, n+1, \dots, 2n\}| = n+1$.
- if $x \in \mathbb{R}$, then

$$|\{y \in \mathbb{R} : y^2 = x\}| = \begin{cases} 2 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < 0. \end{cases}$$

If A is an infinite set, then we write $|A| = \infty$.
(In fact, there is a definition of cardinality for infinite sets, under which different infinite sets can have different cardinalities. But we won't explore this here.)

5.6 Counting subsets

Defn: If X is a set, the <u>power set</u> of X is the set of all subsets of X. We write this as $\mathcal{P}(X)$:

$$\mathcal{P}(X) = \{S : S \subseteq X\}.$$

e.g. • $\mathcal{P}(\{1,2,6\}) = \{\emptyset, \{1\}, \{2\}, \{6\}, \{1,2\}, \{1,6\}, \{2,6\}, \{1,2,6\}\}$. 8

• $\mathcal{P}(\{5\}) = \{\emptyset, \{5\}\}$. 2

• $\mathcal{P}(\emptyset) = \{\emptyset\}$ 1

Be careful! $\emptyset$ and $\{\emptyset\}$ are not the same thing.

$\emptyset$ is the set with no elements

$\{\emptyset\}$ is the set with one element, namely $\emptyset$.

Question: If $|A| = n$, then what is $|\mathcal{P}(A)|$?

<u>Counting by making a series of choices</u>

Primary school example: I have 3 bags, each containing 4 apples. How many apples are there? Answer: 3×4.

We can think about the number of apples as the number of ways to <u>choose</u> an apple. We can choose an apple in two stages:

- choose a bag (3 options)
- choose an apple from that bag (4 options).

Multiplication Principle: Suppose we want to choose an object X. Suppose we can break the choosing process down into steps. Suppose that <u>the number of options at each step does not depend on which options we chose at earlier steps</u>. Then the total number of possibilities for X is the product of the number of options at each stage.

e.g. • How many permutations of $1, 2, \ldots, n$ are there?
if $n=3$, there are six: 123, 132, 213, 231, 312, 321.

To choose a permutation $a_1 a_2 \ldots a_n$, we make a series of choices:

- choose a_1 (n options)

- choose a_2 ($n-1$ options: a_2 can be any number except a_1)
- choose a_3 ($n-2$ options)

  ⋮

- choose a_n (1 option).

So by the Multiplication Principle, the number of permutations is $n \times (n-1) \times (n-2) \times \dots \times 1 = n!$

- How many games are played in the Premiership each season? (There are 20 teams. Each team plays each other team home and away.)

  To choose a game, we can:
  - choose the home team (20 options)
  - choose the away team (19 options)

  So the number of games is $20 \times 19 = 380$.

Theorem 5.2: Suppose X is a finite set, and $|X| = n$. Then $|\mathcal{P}(X)| = 2^n$.

Proof: We can write the elements of X as $x_1, x_2, \dots, x_n$.

To choose a subset $S \subseteq X$, we make a sequence of choices:

- choose whether $x_1 \in S$ or not (2 options)
- choose whether $x_2 \in S$ or not (2 options)

  ⋮

- choose whether $x_n \in S$ or not (2 options).

So by the Multiplication Principle, the number of ways to choose a subset is $2 \times 2 \times \dots \times 2 = 2^n$. $\square$

5.7 Counting subsets of a particular size

Def^n: Suppose X is a set and $k \in \mathbb{Z}$. A <u>k-element subset</u>

of X means a subset S of X with exactly k elements.
If $|X| = n$, then we write $\binom{n}{k}$ for the number of k-element subsets of X.

$\binom{n}{k}$ is pronounced "n choose k", and is called a <u>binomial coefficient</u>.

$\cancel{{}^{n}C_{k}}$

e.g.
- the subsets of $\{1,2,3\}$ are
  $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$.

  So $\binom{3}{0} = 1$ $\binom{3}{1} = 3$ $\binom{3}{2} = 3$ $\binom{3}{3} = 1$,
  $\binom{3}{k} = 0$ for any other k.
- $\binom{5}{2} = 10$: the 2-element subsets of $\{1,2,3,4,5\}$ are $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}$
  $\{2,3\}, \{2,4\}, \{2,5\}$,
  $\{3,4\}, \{3,5\}$,
  $\{4,5\}$.

- if $k < 0$ or $k > n$, then $\binom{n}{k} = 0$
- $\binom{n}{0} = 1$: the only 0-element subset of X is $\emptyset$.
- $\binom{n}{n} = 1$.
- $\binom{n}{1} = n$.
- $\binom{n}{k} = \binom{n}{n-k}$: choosing k elements to be in a k-element subset S is equivalent to choosing $n-k$ elements to not be in S.

Theorem 5.3: Suppose $n, k \in \mathbb{Z}$ and $0 \leq k \leq n$. Then

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}.$$

Proof: We want to count the ways of choosing a k-element subset $\{a_1, a_2, \dots, a_k\}$ of $\{1, 2, \dots, n\}$.

First we count the ways of choosing k different elements $a_1, a_2, \dots, a_k$ <u>in order</u>:

- choose a_1 (n options)
- choose a_2 ($n-1$ options: a_2 can be anything except a_1)
- choose a_3 ($n-2$ options)

  $\vdots$

- choose a_k ($n-k+1$ options).

So by the Multiplication Principle, the number of ways of choosing $a_1, \dots, a_k$ in order is

$n \times (n-1) \times \dots \times (n-k+1)$

$$= \frac{n \times (n-1) \times \dots \times (n-k+1) \times (n-k) \times (n-k-1) \times \dots \times 1}{(n-k) \times (n-k-1) \times \dots \times 1}$$

$$= \frac{n!}{(n-k)!}.$$

Having chosen $a_1, a_2, \dots, a_k$ in order, we obtain a k-element subset $\{a_1, a_2, \dots, a_k\}$.

How many times have we counted each k-element subset $\{a_1, a_2, \dots, a_k\}$? The k elements can be ordered in $k!$ ways, so we have counted each k-element subset $k!$ times.

So we can just divide by $k!$ to get the number of k-element subsets as

$$\frac{n!}{(n-k)!\,k!}$$ $\square$

e.g. To find $\binom{4}{2}$, we first count the ways of choosing a_1, a_2 in order:

1, 2	1, 3	1, 4
2, 1	2, 3	2, 4
3, 1	3, 2	3, 4
4, 1	4, 2	4, 3

$\to$ 12 options $= \frac{4!}{2!}$

But now we've counted each subset $\{a_1, a_2\}$ twice: as a_1, a_2, and also a_2, a_1

So we divide by 2 to get $\binom{4}{2} = \frac{12}{2} = 6$.

e.g.

- $\binom{n}{0} = \frac{n!}{n!\,0!} = \frac{1}{0!} = 1.$
- $\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = \frac{n}{1!} = n.$
- $\binom{n}{n-k} = \frac{n!}{(n-(n-k))!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$
- what is $\binom{11}{5}$?

$$\binom{11}{5} = \frac{11!}{6!\,5!} = \frac{2\times 3\times 11\times \cancel{10}\times \cancel{9}\times \cancel{8}\times 7\times \cancel{6}\times \cancel{5\times 4\times 3\times 2\times 1}}{\cancel{6}\times \cancel{5}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times 1\times \cancel{5\times 4\times 3\times 2\times 1}}$$

$$= 2\times 3\times 11\times 7 = 462.$$

6. Functions

6.1 Definitions

Def^n: Suppose A and B are sets. A <u>function</u> from A to B is a rule for assigning an element of B to each element of A.

We write $f : A \to B$ to mean "f is a function from A to B". If $a \in A$, we write $f(a)$ for the element of B assigned to a by f. We write $a \mapsto b$ to mean $f(a) = b$.

$f(a)$ is often called the <u>value</u> of f at a. A is called the <u>domain</u> of f, and B is the <u>codomain</u> of f.

Arrows: use $\to$ between sets: $f : A \to B$

use $\mapsto$ between elements: $a \mapsto b$.

Examples:

- $f: \mathbb{R} \to \mathbb{R}$ defined by $\underbrace{x \mapsto x^2}_{\text{means } f(x) = x^2}$.

- $f: \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto n^2$.

  (The last two examples are not the same function: the domain and codomain are part of the definition of a function.)

- $f: \mathbb{N} \to \mathbb{Z}$ defined by $n \mapsto n-1$.

- $f: \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 2 \\ 2 & \text{if } x > 2. \end{cases}$$

  (This is a function with a "piecewise" definition, where a different rule applies on different parts of the domain. When defining functions like this, make sure that $f(x)$ is uniquely specified.)

- $f: \{1, 2, 3\} \to \{\text{red}, \text{blue}\}$ defined by

  $1 \mapsto \text{red}$, $2 \mapsto \text{blue}$, $3 \mapsto \text{blue}$.

- If X is a finite set, define

$$f: \mathcal{P}(X) \to \mathbb{Z}$$

  by $A \mapsto |A|$.

- $f: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by

  $A \mapsto \mathbb{N} \setminus A$.