Proposition 5.1: Suppose A, B, C are sets. Then:

(a) $(A \cap B) \cap C = A \cap (B \cap C)$.

(b) $(A \cup B) \cup C = A \cup (B \cup C)$.

(c) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof of parts of Proposition 5.1: We'll prove parts a, c, d.

(a) If $x \in (A \cap B) \cap C$, then

$x \in A \cap B$ and $x \in C$,

which means that

$x \in A$ and $x \in B$ and $x \in C$.

If $x \in A \cap (B \cap C)$, then

$x \in A$ and $x \in B \cap C$,

which means that

$x \in A$ and $x \in B$ and $x \in C$.

So the condition for x to be an element of $(A \cap B) \cap C$ is the same as the condition for x to be an element of $A \cap (B \cap C)$. So these two sets have the same elements, so they're equal.

(c) $x \in (A \triangle B) \triangle C$ means

$x \in A \triangle B$ and $x \notin C$, <u>or</u>

$x \notin A \triangle B$ and $x \in C$.

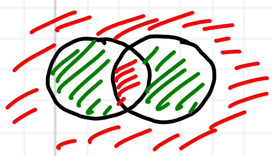
This is the same as

$x \in A$ and $x \notin B$ and $x \notin C$, or

$x \notin A$ and $x \in B$ and $x \notin C$, or

$x \in A$ and $x \in B$ and $x \in C$, or

$x \notin A$ and $x \notin B$ and $x \in C$.



So $x \in (A \triangle B) \triangle C$ means that x belongs to exactly one of A, B and C, or all three of A, B and C.

The same is true for $A \triangle (B \triangle C)$, so

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

(d) We'll show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
and $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$.

Suppose first that $x \in A \cup (B \cap C)$. Then
$x \in A$ or $x \in B \cap C$. Let's consider two cases:

- if $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$
  so $x \in (A \cup B) \cap (A \cup C)$.
- if $x \in B \cup C$, then $x \in B$ and $x \in C$, so
  $x \in A \cup B$ and $x \in A \cup C$,
  so $x \in (A \cup B) \cap (A \cup C)$.

For the other direction. Suppose $x \in (A \cup B) \cap (A \cup C)$.
Then $x \in A \cup B$ and $x \in A \cup C$. We consider
two cases, according to whether $x \in A$ or not.

- if $x \in A$, then $x \in A \cup (B \cap C)$.
- if $x \notin A$, then $x \in B$ (because $x \in A \cup B$)
  and $x \in C$ (because $x \in A \cup C$),
  so $x \in (B \cap C)$, so $x \in A \cup (B \cap C)$. $\square$

Because of Proposition 5.1, we can write expressions like

$A \cap B \cap C$

$A \cup B \cup C$

$A \Delta B \Delta C$

without brackets. But for some expressions we need
brackets: $(A \cap B) \cup C$ is not the same as $A \cap (B \cup C)$.

e.g. $A = \{3, 5, 7\}$

$B = \{2, 4, 12\}$

$C = \{9, 15, 21\}$

Then $(A \cap B) \cup C = \{9, 15, 21\}$.
But $A \cap (B \cup C) = \{3, 5, 7\} \cap \{2, 4, 9, 12, 15, 21\} = \emptyset$.

(Side exercise: is it always true that
$(A \cap B) \cup C \subseteq A \cap (B \cup C)$
or $(A \cap B) \cup C \supseteq A \cap (B \cup C)$?)

So we sometimes need brackets with set operations.
(There is no BIDMAS rule.)

5.4 Another set operation: Cartesian product

If a and b are objects, then we write (a,b) for the ordered pair "a then b".

This is not the same as the set $\{a,b\}$. The order matters: $(1,3) \neq (3,1)$.

We can have $a=b$: $(0,0)$ is an ordered pair (not the set $\{0\}$).

Definition: Suppose A and B are sets. The <u>Cartesian product</u> is the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$.

We write $A \times B$ for the Cartesian product of A and B.

$$A \times B = \{(a,b) : a \in A,\ b \in B\}.$$

e.g.

- $\{1,2,3\} \times \{2,4\} = \{(1,2), (1,4), (2,2), (2,4), (3,2), (3,4)\}$.
- $\{2,4\} \times \{1,2,3\} = \{(2,1), (4,1), (2,2), (4,2), (2,3), (4,3)\}$.
- $\mathbb{R} \times \mathbb{R} = \{(x,y) : x,y \in \mathbb{R}\}$ can be regarded as the plane, by using Cartesian coordinates.
- If S is any set, then
  $S \times \emptyset = \emptyset$: an element of $S \times \emptyset$ is a pair (s,t) where $s \in S$ and $t \in \emptyset$. But there are no such t, so there are no such pairs.