Theorem 4.8: Suppose $a, b \in \mathbb{N}$. Then
$$\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}.$$

Pf (non-examinable): Let $g = \gcd(a,b)$ and $m = \operatorname{lcm}(a,b)$.

Then $g \mid b$, so $\frac{b}{g} \in \mathbb{N}$. So

$\frac{ab}{g} = a \times \frac{b}{g}$ is a natural number divisible by a.

Similarly, $\frac{ab}{g}$ is divisible by b.

So $\frac{ab}{g}$ is a common multiple of a and b. So by Lemma 4.7 $\frac{ab}{g}$ is a multiple of m. So

$\frac{ab}{g} = km$ for some $k \in \mathbb{N}$. We need to show that $k=1$.

We know $a \mid m$, so $\frac{m}{a} \in \mathbb{N}$. So

$b = \frac{m}{a} \times kg$ is a natural number divisible by kg.

Similarly, a is divisible by kg.

So kg is a common divisor of a and b. But g is the greatest common divisor of a and b, so $kg \leq g$.

Since $k \in \mathbb{N}$, this means $k = 1$. $\square$

e.g. What is $\operatorname{lcm}(68, 20)$?

We saw earlier that $\gcd(68, 20) = 4$.

So $\operatorname{lcm}(68,20) = \frac{68 \times 20}{4} = 68 \times 5 = 340.$

What is $\operatorname{lcm}(2904, 1001)$?

We saw earlier that $\gcd(2904, 1001) = 11$.

So $\operatorname{lcm}(2904, 1001) = \frac{2904 \times 1001}{11} = 264 \times 1001$

$= 264264.$

5. Sets

5.1 Definition, notation and examples

A **set** means a collection of objects gathered together. (The objects could be numbers, functions, sets, statements, vegetables, ...)

If A is a set, the objects in A are called the **elements** of A.

Notation: we write $x \in A$ to mean "x is an element of A", and $x \notin A$ to mean "x is not an element of A".

Writing sets

The simplest way to write a set is to write the elements in a list, separated by commas, and enclosed by $\{ \}$.

e.g. $\{1, 2, 3\}$ is the set whose elements are 1, 2 and 3, and nothing else.

The order doesn't matter:

$$\{1, 2, 3\} = \{2, 3, 1\}.$$

We can write elements more than once (repeated elements are ignored):

$$\{1, 2, 3, 2\} = \{1, 2, 3\}.$$

Important: A set is defined by its elements. So if two sets have the same elements, they are the same set.

More ways to write sets

(1) For large sets (or even infinite sets) we can use ... to indicate a list of elements that follows a pattern.

$\{1, 3, 5, 7, \dots\}$ is the set of odd natural numbers

$\{1, 3, 5, 7, \dots, 999\}$ is the set of odd natural numbers from 1 to 999.

$\{\dots, -4, -2, 0, 2, 4, \dots\}$ is the set of even integers.

We can use this notation with variables: if $n \in \mathbb{N}$, then

$\{1, 2, 3, \dots, n\}$ means the set of natural numbers from 1 to n.

(2) Some sets can be described in words:

Let S be the set of odd natural numbers.
Let T be the set of all cats in France.

(3) Some sets (especially sets of numbers) have special agreed notation:

$\mathbb{N}$ is the set of natural numbers

$\mathbb{Z}$ is the set of integers

$\left.\begin{array}{l} \mathbb{Q} \text{ is the set of rational numbers} \\ \mathbb{R} \text{ is the set of real numbers} \\ \mathbb{C} \text{ is the set of complex numbers} \end{array}\right\}$ more on these later in the module

$\emptyset$ is the <u>empty set</u>, i.e. the set with no elements.

(4) We can define a set to be the set of all objects that satisfy given conditions:

e.g. $\{x : x \in \mathbb{Q}, x > 0\}$ is the set of all positive rational numbers.

Notation in general:

$$\{ \quad : \quad \}$$

objects in the set

conditions those objects must satisfy.

(Sometimes $|$ is used instead of $:$)

More examples:

$\{x : x \in \mathbb{R},\ a \leq x \leq b\}$ is the interval $[a,b]$.

(x is another kind of dummy variable: $\{y : y \in \mathbb{R},\ a \leq y \leq b\}$ is the same set.)

$\{x : x \in \mathbb{N},\ x = 2k$ for some $k \in \mathbb{N}\}$ is the set of even natural numbers.
(Here we've introduced another dummy variable k with a quantifier "for some"!)

Very often one of the conditions on x is that it belongs to some other set S. In this case we often put "$\in S$" before the colon.

e.g. $\{x \in \mathbb{Z} : x \geq 0\}$ is the set of non-negative integers

$\{d \in \mathbb{N} : d \mid 10\}$ is the set $\{1, 2, 5, 10\}$.

$\{d \in \mathbb{N} : 10 \mid d\}$ is the set $\{10, 20, 30, \ldots\}$.

(5) A more advanced version of (4): we can take all values of the variable x that satisfy given conditions, and then apply an operation to all of them.

e.g. The set of even natural numbers can be written as

$\{2x : x \in \mathbb{N}\}$.

- The set of odd natural numbers is $\{2x-1 : x \in \mathbb{N}\}$.
- $\{x^2 : x \in \mathbb{Z}\}$ is the set of all squares, i.e. $\{0, 1, 4, 9, \dots\}$.
- $\{x^2 : x \in \mathbb{R}\}$ is the set of all non-negative real numbers, i.e. $\{x \in \mathbb{R} : x \geq 0\}$.
- $\{-x : x \in \mathbb{N}\}$ is the set of all negative integers.

We can use more than one variable:

- $\{x - y : x, y \in \mathbb{N}\} = \mathbb{Z}$.
- $\{xy : x, y \in \mathbb{N},\ x, y > 1\}$ is the set of all composite natural numbers.
- $\mathbb{Q} = \{\frac{x}{y} : x, y \in \mathbb{Z},\ y \neq 0\}$.

Some more examples:

- Suppose $a \in \mathbb{Z}$. Then the set

  $\{2x + a : x \in \mathbb{Z}\}$ depends on a.

  It is the set of even integers if a is even or the set of odd integers if a is odd.
- $\{n \in \mathbb{N} : n - 1 \in \mathbb{N}\} = \{2, 3, 4, \dots\}$.
- $\{n \in \mathbb{N} : \underbrace{n + 1 \in \mathbb{N}}_{\text{redundant condition}}\} = \mathbb{N}$
- $\{n \in \mathbb{N} : n < 0\} = \emptyset$.

5.2 Subsets

Definition: Suppose A and B are sets. A is a subset of B if every element of A is an element of B. A is a proper subset of B if A is a subset of B and $A \neq B$.

Notation: we write $A \subseteq B$ to mean "A is a subset of B".

we write $A \subset B$ to mean "A is a proper subset of B".

e.g.
- $\{2,3\} \subseteq \{1,2,3\}$.
- if $m,n \in \mathbb{N}$ and $m \leq n$, then $\{1,2,\dots,m\} \subseteq \{1,2,\dots,n\}$.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- $A \subseteq A$ for every set A.
- $\emptyset \subseteq A$ for every set A.
- $\{x \in \mathbb{Q} : x^2 - x \in \mathbb{Z}\} \subseteq \mathbb{Q}$.

In general, if S is a set, then any set of the form
$\{x \in S :$ some conditions$\}$
is a subset of S.

5.3 Set operations

We use operations to create new sets from existing sets. Suppose A and B are sets.

The <u>union</u> $A \cup B$ is the set of all objects that are in A or B or both.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The <u>intersection</u> $A \cap B$ is the set of all objects that are in A and in B.

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

This can also be written as
$\{x \in A : x \in B\}$
or $\{x \in B : x \in A\}$.

The *difference* $A \setminus B$ is the set of all elements of A that are not elements of B

$$A \setminus B = \{x \in A : x \notin B\}.$$

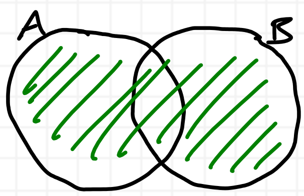
The *symmetric difference* $A \Delta B$ is the set of all objects that are in A or B but not both.

$$A \Delta B = \{x : (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$$

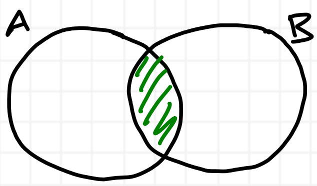
$$= (A \cup B) \setminus (A \cap B)$$

$$= (A \setminus B) \cup (B \setminus A).$$

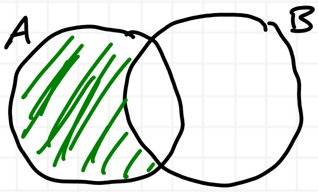
Often these operations are illustrated using Venn diagrams.



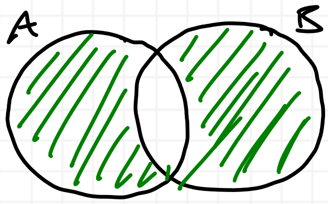
$A \cup B$



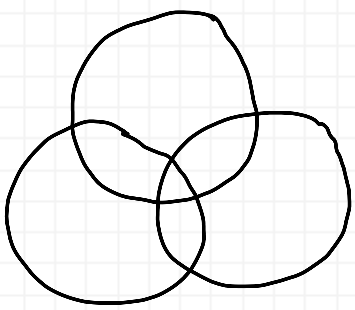
$A \cap B$



$A \setminus B$



$A \Delta B$



Venn diagrams are very useful for visualising the interactions between two or three sets. But for more than three sets, we need to get used to working with elements.

Def^n: Two sets A and B are disjoint if $A \cap B = \emptyset$.

Now we'll prove some identities concerning set operations.

Proposition 5.1: Suppose A, B, C are sets. Then:

(a) $(A \cap B) \cap C = A \cap (B \cap C)$.

(b) $(A \cup B) \cup C = A \cup (B \cup C)$.

(c) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The equations in Prop^n 5.1 all say that two sets are equal. A standard way to prove that two sets S, T are equal is to show that $S \subseteq T$ and $T \subseteq S$. i.e. every element of S is an element of T, and vice versa.