Here's another example.

Theorem 3.8: There is no smallest positive rational number.

Pf: We use proof by contradiction. Suppose there is a smallest positive rational number r.

Then $r > 0$.

So $\frac{r}{2} > 0$.

Adding $\frac{r}{2}$ to both sides, we get

$$r > \frac{r}{2}.$$

So $r > \frac{r}{2} > 0$. $\frac{r}{2}$ is rational because r is,

so $\frac{r}{2}$ is a smaller positive rational number than r.

So r is not the smallest. ↯ ※

So there is no such r. □

3.7 Special Technique 3: proof by induction

Proof by induction is used when the statement being proved involves a positive integer n.

Suppose we want to prove the statement $P(n)$ for every positive integer n. To prove it by induction, we do two things:

Base case: prove that $P(1)$ is true.

Inductive step: prove that $P(n-1) \Rightarrow P(n)$ for every $n \geq 2$.

Here's an example.

Theorem 3.9: Suppose n is a positive integer. Then

$$\sum_{k=1}^{n} 2k-1 = n^2.$$

This theorem says

$$\begin{aligned} 1 &= 1^2 \quad ✓ \\ &\Downarrow \\ 1+3 &= 2^2 \\ &\Downarrow \\ 1+3+5 &= 3^2 \\ &\Downarrow \\ 1+3+5+7 &= 4^2 \end{aligned}$$

Base case

Inductive step

⋮

Proof: We use proof by induction. Let $P(n)$ denote the equation $\sum_{k=1}^{n} 2k-1 = n^2$.

Base case: $P(1)$ says $1 = 1^2$, which is true.

Inductive step: Assume $n \geq 2$ and $P(n-1)$ is true.
Then $\sum_{k=1}^{n-1} 2k-1 = (n-1)^2$.

So
$$\sum_{k=1}^{n} 2k-1 = \sum_{k=1}^{n-1} 2k-1 + 2n-1$$
(splitting the last term from the sum)
$$= (n-1)^2 + 2n-1$$
(using $P(n-1)$)
$$= n^2-2n+1 + 2n-1$$
$$= n^2$$

so $P(n)$ is true.

So $P(n)$ is true for all n. $\square$

Terminology: The assumption that $P(n-1)$ is true in the inductive step is called the <u>inductive hypothesis</u>.

Another example:

Theorem 3.10: Suppose n is a positive integer. Then $8^n - 3^n$ is divisible by 5.

Pf: We use proof by induction. Let $P(n)$ denote the statement "$8^n - 3^n$ is divisible by 5".

Base case: $P(1)$ says $8-3$ is divisible by 5, which is true.

Inductive step: Suppose $n \geq 2$ and $P(n-1)$ is true. Then

$8^{n-1} - 3^{n-1} = 5k$ for some integer k.

So

$$
\begin{aligned}
8^n - 3^n &= 8^{n-1} \times 8 - 3^{n-1} \times 3 \\
&= (5k + 3^{n-1}) \times 8 - 3^{n-1} \times 3 \quad \text{(using } P(n-1)\text{)} \\
&= 40k + 3^{n-1} \times (8-3) \\
&= 5 \times (8k + 3^{n-1})
\end{aligned}
$$

which is divisible by 5. So $P(n)$ is true.

So $P(n)$ is true for all n. $\square$

Tips for writing induction proofs:

- it's <u>very</u> helpful to give a name (like $P(n)$) to the statement being proved.
- don't confuse a statement with a number. e.g. in Theorem 3.9, $P(n)$ is <u>not</u> the number $\sum_{k=1}^{n} 2k-1$. It's the statement that two numbers are equal.
- explain how to get from one line to the next (unless it's a simple algebraic manipulation). In particular, <u>say</u> where $P(n-1)$ is being used.

Why does proof by induction work? We want to prove $P(1)$, $P(2)$, $P(3)$, ...

$P(1)$ is true by the base case.

$P(2)$ is true because $P(1)$ is true and $P(1) \Rightarrow P(2)$

$P(3)$ is true because $P(2)$ is true and $P(2) \Rightarrow P(3)$

....

Proof by strong induction works as follows:

Base case: prove that $P(1)$ is true.

Inductive step: prove that if $n \geq 2$ and $P(1), P(2), \ldots, P(n-1)$ are all true, then

$P(n)$ is true.

Warning about the base case: Sometimes the base case is not $P(1)$. e.g. the theorem might apply to all non-negative integers n. In which case $P(0)$ is the base case.

Sometimes we need more than one base case. e.g. we might prove $P(1)$ and $P(2)$ in the base case, and start the inductive step at $n = 3$.

Tip: Write the inductive step first. Then the base cases are any cases not covered by the inductive step.

We'll do two examples using the Fibonacci sequence: $F_1, F_2, F_3, \ldots$ defined by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

$$1, 1, 2, 3, 5, 8, \ldots$$

Theorem 3.11: Suppose n is a positive integer. Then $F_n < 2^n$.

Pf: We use strong induction. Let $P(n)$ denote the inequality $F_n < 2^n$.

Base case: $P(1)$ says $F_1 < 2^1$, which is true.
$P(2)$ says $F_2 < 2^2$, which is true.

Inductive step: Suppose $n \geq 3$, and that $P(1), \ldots, P(n-1)$ are all true. Then

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &< 2^{n-1} + 2^{n-2} \quad \text{(using } P(n-1) \text{ and } P(n-2)\text{)} \\ &< 2^{n-1} + 2^{n-1} \\ &= 2^n \end{aligned}$$

needs strong induction

so $P(n)$ is true.

So $P(n)$ is true for all n. $\square$

Another example:

Theorem 3.12: Suppose n is a positive integer. Then F_n is even iff n is divisible by 3.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

Pf: We use strong induction. Let $P(n)$ denote the statement in the theorem.

Base cases: $P(1)$ and $P(2)$ are true, because F_1 and F_2 are both odd, and neither 1 nor 2 is divisible by 3.

Inductive step: Suppose $n \geq 3$ and $P(1), \dots, P(n-1)$ are all true. We consider two cases.

Case 1: Suppose n is divisible by 3. Then neither $n-1$ nor $n-2$ is divisible by 3. So by $P(n-1)$ and $P(n-2)$, F_{n-1} and F_{n-2} are both odd. So

$$F_n = F_{n-1} + F_{n-2} \quad \text{is even.}$$

Case 2: Suppose n is not divisible by 3. Then exactly one of $n-1$ and $n-2$ is divisible by 3. So (using $P(n-1)$ and $P(n-2)$) exactly one of F_{n-1} and F_{n-2} is even (and the other is odd). So $F_n = F_{n-1} + F_{n-2}$ is odd.

So $P(n)$ is true.

So $P(n)$ is true for all n. □

3.8 Finding mistakes in proofs

When you write a proof, you should check to see whether it makes sense and proves the thing it's supposed to prove. Here are some tips:

- Make sure the "grammar" of the proof makes sense. If n is a number and f is a function, don't write $f = n$. If $P(n)$ is a statement, don't write $P(n) = n^2$.

- Make sure the variables only take the values they're supposed to. If the theorem says n is a positive integer, don't consider the case $n = 0$ or $n = \frac{1}{2}$.

- Make sure the proof goes in the right direction. Start from what you know, and end with what you're trying to prove.

    Exceptions:

    → for a proof using the contrapositive, start from (not Q), and end with (not P).

    → for a proof by contradiction, start by assuming the conclusion is false and end with a contradiction.

- Check that the proof uses all the hypotheses. If "n is prime" is one of the hypotheses but it doesn't appear in the proof, then either the proof is wrong, or the hypothesis is not needed: check whether the theorem is plausible when n is not prime.

- Check the proof for special values of the variables. e.g. if the proof includes variables m and n satisfying $1 \leq m \leq n$, then check through the proof for $m = 1$ and $m = n$.