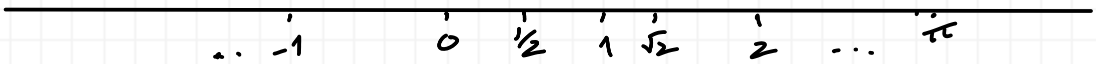
Defn: Let $z = a+bi$, where $a, b \in \mathbb{R}$. The complex <u>conjugate</u> of z is $a - bi$. We write this as $\bar{z}$.

We saw above that $z\bar{z} = a^2+b^2$, which is real. Also, $z + \bar{z} = 2a$, which is real. In fact, $\bar{z}$ is the only complex number such that $z\bar{z}$ and $z+\bar{z}$ are both real.

9.2 The complex plane

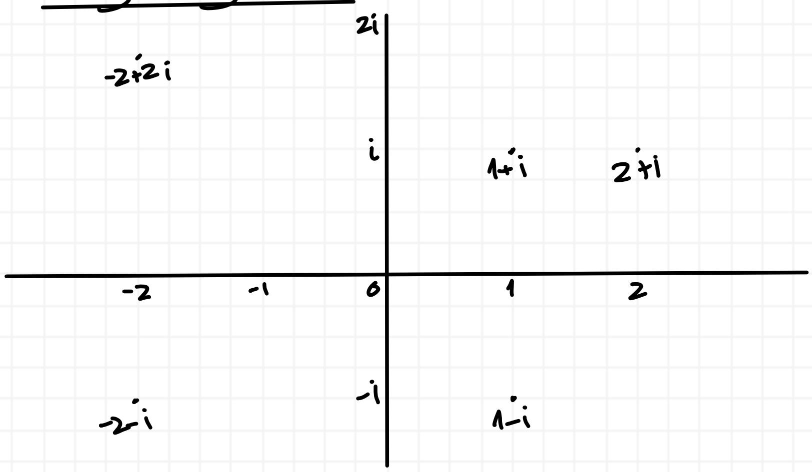
We visualise $\mathbb{R}$ as a number line:



How do we visualise $\mathbb{C}$? Where does i fit in this picture?

The <u>complex plane</u> means the plane $\mathbb{R}^2$, with each point (a,b) representing the complex number $a+bi$.

The x-axis is now called the <u>real axis</u>, and the y-axis is the <u>imaginary axis</u>.



Going from a line to a plane means that we lose the ordering $<$: there is no ordering $<$ on $\mathbb{C}$ satisfying the familiar rules. In particular, in $\mathbb{R}$ we have the rules:

if $a<b$ and $c>0$, then $ac<bc$

if $a<b$ and $c<0$, then $ac>bc$.

Suppose these rules hold in $\mathbb{C}$. Is $i>0$ or $i<0$?

if $i>0$, then

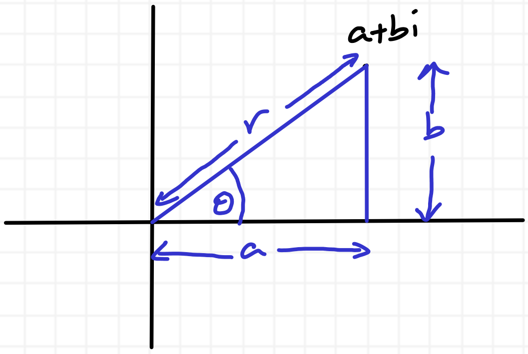
$0<i$ and $i>0$, so $0 \times i < i \times i$, so $0<-1$ ☹

if $i<0$, then

$i<0$ and $i<0$, so $i \times i > 0 \times i$, so $-1>0$ ☹.

In particular, if a complex number has two square roots, we can't talk about "the positive square root".

The complex plane gives another way to write complex numbers. Consider $z = a+bi \in \mathbb{C}$:

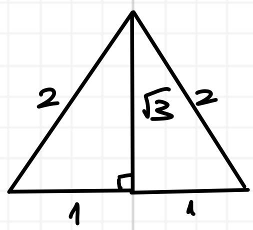


Let r be the distance from z to 0, and θ the anticlockwise angle from the real axis to the line joining z to 0.

Then
$$r = \sqrt{a^2+b^2} = z\bar{z}$$
$$\theta = \tan^{-1}\left(\tfrac{b}{a}\right)$$
$$a = r\cos\theta$$
$$b = r\sin\theta.$$

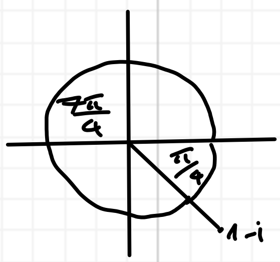
So we can write z as $r(\cos\theta + i\sin\theta)$. This is called the <u>polar form</u> of z.

e.g. $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$1-i = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$



$i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$-3 = 3(\cos\pi + i\sin\pi)$

Defn: Let $z = a+bi$, where $a, b \in \mathbb{R}$.
The <u>modulus</u> of z is $\sqrt{a^2+b^2}$, written $|z|$

The <u>argument</u> of z is the angle $\theta \in [0, 2\pi)$ such that $|z|\cos\theta = a$, $|z|\sin\theta = b$. This is written as $\arg(z)$.

Remarks:

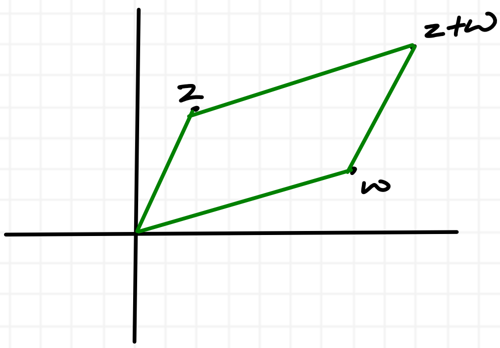
- Some people take $\arg(z) \in (-\pi, \pi]$.
- $\arg(0)$ is undefined.
- $\arg(a+bi)$ is essentially $\tan^{-1}(b/a)$. But precisely:

$$\arg(a+bi) = \begin{cases} \tan^{-1}(b/a) & \text{if } a \neq 0 \\ \pi/2 & \text{if } a = 0 < b \\ 3\pi/2 & \text{if } a = 0 > b. \end{cases}$$

Adding complex numbers works like adding vectors in the plane:

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$
$$(a,c) + (b,d) = (a+c, b+d).$$

So geometrically, the complex numbers $z, w, 0, z+w$ make a parallelogram:



To multiply in the complex plane, it's helpful to use polar form.

Suppose
$$z = r(\cos\alpha + i\sin\alpha)$$
$$w = s(\cos\beta + i\sin\beta)$$

Then
$$zw = rs(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= rs((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta))$$
$$= rs(\cos(\alpha+\beta) + i\sin(\alpha+\beta)).$$

in polar form!

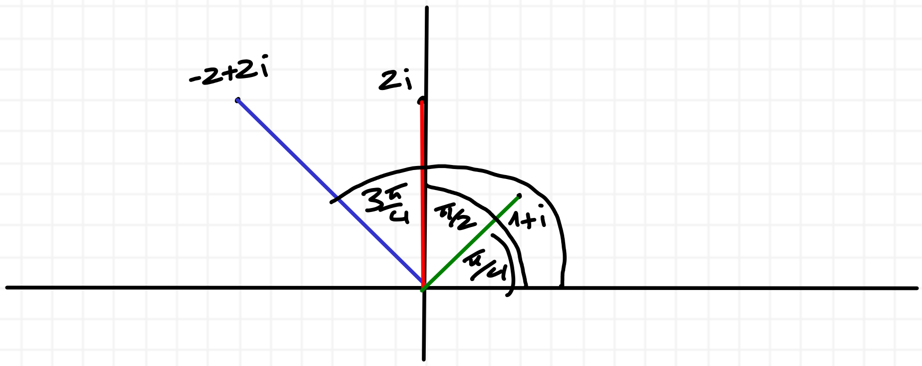
So, to multiply complex numbers in polar form, we just multiply their moduli and add their arguments.

e.g. $z = 1+i = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$w = 2i = 2\,(\cos\pi/2 + i\sin\pi/2)$.

Then $zw = (1+i)(2i) = -2+2i$.

$= 2\sqrt{2}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$.

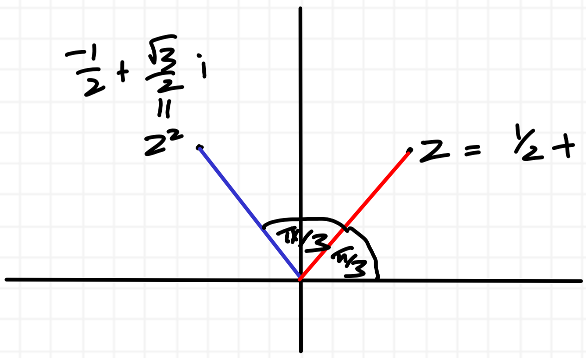


e.g $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.

So $z^2 = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^2 = \frac{1}{4} - \frac{3}{4} + (\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4})i$

$= -1/2 + \frac{\sqrt{3}}{2}i$.

$= (\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$



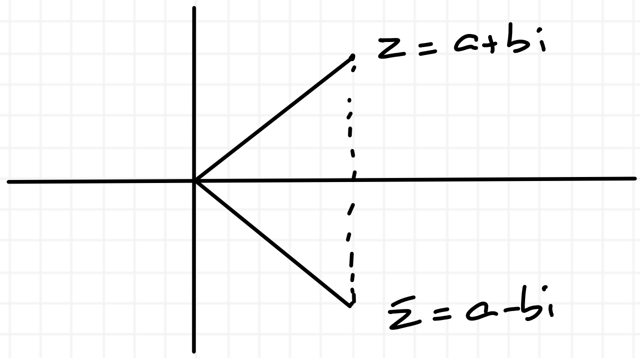
Multiplying in polar form shows that every complex number has a square root in $\mathbb{C}$: if

$z = r(\cos\theta + i\sin\theta)$, then let

$w = \sqrt{r}\,(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$. Then $w^2 = z$.

The complex conjugate is also easy to visualise in the complex plane:

if $z = a+bi$, then $\bar{z} = a-bi$, which is obtained by reflecting in the real axis:



if $z = r(\cos\theta + i\sin\theta)$, then

$$\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$$

Multiplying in polar form gives a formula for powers of complex numbers.

Theorem 9.1 (De Moivre's Theorem): If $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$, then $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Pf: We use proof by induction. Let $P(n)$ denote the equation in the theorem.

Base case: $P(1)$ says

$$\cos\theta + i\sin\theta = \cos\theta + i\sin\theta$$

which is true.

Inductive step: Suppose $n \geq 2$ and $P(n-1)$ is true.

Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{n-1} \times (\cos\theta + i\sin\theta) \\ &= \cos((n-1)\theta) + i\sin((n-1)\theta) \times (\cos\theta + i\sin\theta) \\ &\quad \text{(using } P(n-1)) \\ &= \cos(n\theta) + i\sin(n\theta) \\ &\quad \text{(using the rule for multiplying in polar form)}\end{aligned}$$

So $P(n)$ is true.

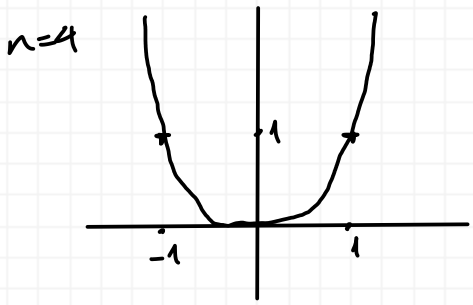
So $P(n)$ is true for all n. $\square$

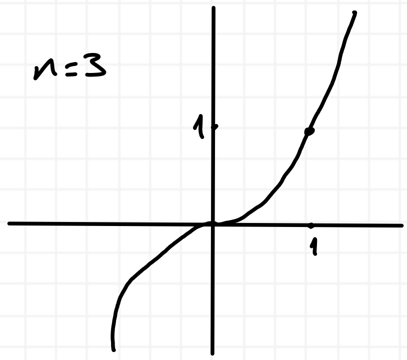
9.3 Roots of unity

("Unity" is a fancy maths word for the number 1.)

Question: given $n \in \mathbb{N}$, what are the solutions to the equation $z^n = 1$?

In $\mathbb{R}$, this is easy: $z = 1$ is a solution, and $z = -1$ is a solution whenever n is even.





But in $\mathbb{C}$, there are more solutions. We can find the solutions using De Moivre's Theorem.

Given $z \in \mathbb{C}$, write z in polar form:

$$z = r(\cos\theta + i\sin\theta).$$

Then, by De Moivre's Theorem,

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

When does this equal 1?

We need $r^n = 1$.

$\arg(z^n) = n\theta$ (may need to subtract a multiple of 2π).

So to get $z^n = 1$, we need

$r^n = 1$, and $n\theta$ must be a multiple of 2π, say $n\theta = 2\pi m$, where $m \in \mathbb{Z}$.

So $z^n = 1$ iff z has the form

$$\cos\left(\frac{2\pi m}{n}\right) + i\sin\left(\frac{2\pi m}{n}\right) \quad \text{for some } m \in \mathbb{Z}.$$

e.g. $n=4$. What are the solutions to $z^4=1$?

$1, -1, i, -i$ are all solutions.

And in fact these are the only solutions: from above, the solutions are $\cos\left(\frac{\pi m}{2}\right) + i\sin\left(\frac{\pi m}{2}\right)$ for $m \in \mathbb{Z}$.

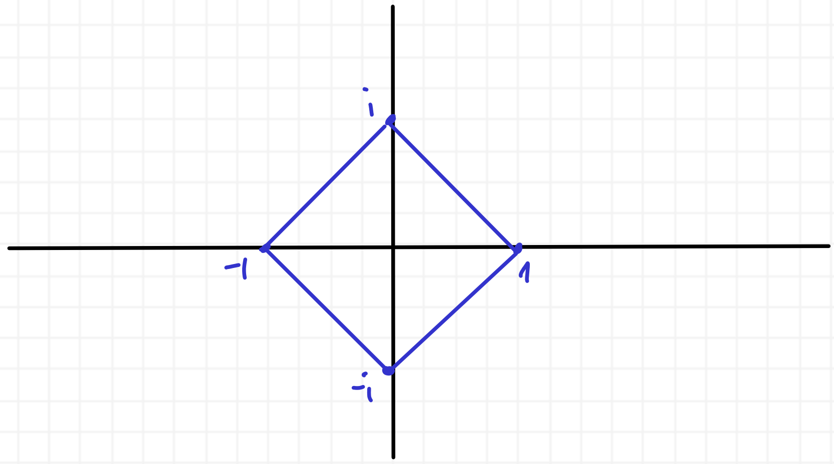
$m=0$ gives $\cos(0) + i\sin(0) = 1$

$m=1$ gives $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

$m=2$ gives $\cos(\pi) + i\sin(\pi) = -1$

$m=3$ gives $\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i$

$m=4$ gives $\cos(2\pi) + i\sin(2\pi) = 1$



$n=3$: the only real solution to $z^3=1$ is $z=1$.

But in general, solutions are

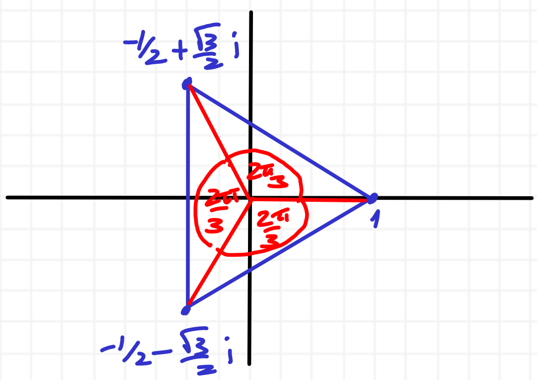
$$\cos\left(\frac{2\pi m}{3}\right) + i\sin\left(\frac{2\pi m}{3}\right)$$

$m=0$: $\cos(0) + i\sin(0) = 1$

$m=1$: $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$m=2$: $\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

$m=3$: $\cos(2\pi) + i\sin(2\pi) = 1$



In general, the equation $z^n = 1$ has exactly n solutions in $\mathbb{C}$. These are all distance 1 from 0, at angles $\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots$. They form a regular n-sided polygon.

In general, any equation of the form

$$z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_1 z + a_0 = 0$$

(where the coefficients $a_{n-1}, a_{n-2}, \ldots, a_0 \in \mathbb{C}$),
always has n solutions in $\mathbb{C}$ (possibly repeated roots).
This is called the Fundamental Theorem of Algebra.