Now we show why we need the real numbers.

Let $X = \{x \in \mathbb{Q} : x^2 \leq 2\}$.

Then X is bounded above, e.g. 2 is an upper bound. X doesn't have a maximum: to show this, we have to show that given any $x \in X$, we can find $y \in X$ such that $y > x$.

Given x, let $y = \frac{2x+2}{x+2}$. We can check that if $x^2 \leq 2$, then $y^2 \leq 2$ and $y > x$.

So X has no maximum.

Also, X does not have a supremum in $\mathbb{Q}$. But it does have a supremum in $\mathbb{R}$, namely $\sqrt{2}$.

This example shows that even if $X \subseteq \mathbb{Q}$, $\sup X$ might be irrational.

Theorem 8.2 (Principle of the supremum): If X is a non-empty subset of $\mathbb{R}$ which is bounded above, then X has a supremum in $\mathbb{R}$.

(This is hard to prove!)

Everything in this section applies with $>$ and $<$ the other way round:

maximum becomes minimum
upper bound $\rightsquigarrow$ lower bound
bounded above $\rightsquigarrow$ bounded below
supremum $\rightsquigarrow$ infimum.

9. Complex numbers

9.1 Definition and operations

Extending $\mathbb{Q}$ to $\mathbb{R}$ allows us to solve equations like $x^2 = 2$, $x^3 = 6$, $x^3 + x^4 = 9$, ... In fact, any equation where we can get close to a solution in $\mathbb{Q}$

we can solve in $\mathbb{R}$.

But there are some equations we can't solve in $\mathbb{R}$. We know that $x^2 \geq 0$ for $x \in \mathbb{R}$. So there is no solution to $x^2 = -1$.

So we extend again, by adding a new number i defined to be a square root of -1.

Defn: A <u>complex number</u> is an expression $a + bi$, where $a, b \in \mathbb{R}$. We write $\mathbb{C}$ for the set of all complex numbers.

If $a \in \mathbb{R}$, then we regard a as an element of $\mathbb{C}$ via $a = a + 0i$. So $\mathbb{R} \subset \mathbb{C}$.

If $z = a + bi$, where $a, b \in \mathbb{R}$, then a is called the real part of z (written $Re(z)$), and b is called the <u>imaginary part</u> of z (written $Im(z)$).

We define addition in $\mathbb{C}$ by adding real and imaginary parts separately:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

e.g. $7 + 4i + 6 - 5i = 13 - i$.

Subtraction works in the same way:

e.g. $7 + 4i - (6 - 5i) = 1 + 9i$.

We define multiplication in $\mathbb{C}$ by multiplying out brackets and using the rule $i^2 = -1$:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$

$$= (ac - bd) + (ad + bc)i.$$

e.g.
- $(4 + 5i)(3 + 2i) = 2 + 23i$
- $(4 - 5i)(3 - 2i) = 2 - 23i$.
- $(4 - 5i) \times 100 = 400 - 500i$.

- $(3+2i)^2 = 5+12i$
- $(4+2i)(2-i) = 10$

  (so the product of two non-real numbers can be real)
- $i \times (-i) = 1$
- $(-i)^2 = -1.$

  (so $-i$ is also a square root of -1).

How do we divide in $\mathbb{C}$? Given two complex numbers z, w, we want $\frac{z}{w}$ to be a complex number such that $\frac{z}{w} \times w = z$.

Dividing by a real number is easy: if $a, b, c \in \mathbb{R}$, then

$$(a+bi) \div c = \frac{a}{c} + \frac{b}{c} i.$$

(But we still can't divide by 0.)

Observation: if $c, d \in \mathbb{R}$, assume $c+di \neq 0$. Then $c^2+d^2 > 0$.

then $(c+di)(c-di) = \underset{\in \mathbb{R}.}{\underset{\uparrow}{c^2+d^2}}$

so $c+di \times \frac{c-di}{c^2+d^2} = 1.$

so $\frac{1}{c+di} = \frac{c-di}{c^2+d^2}.$

So to divide by $c+di$, we can just multiply by $\frac{1}{c+di} = \frac{c-di}{c^2+d^2}.$

So: if $a, b, c, d \in \mathbb{R}$ and $c+di \neq 0$, then

$$(a+bi) \div (c+di) = \frac{(a+bi)(c-di)}{c^2+d^2}.$$

e.g.

- $(4+i) \div (4+3i) = \dfrac{(4+i)(4-3i)}{25}$

  $= \dfrac{19}{25} - \dfrac{8}{25} i.$

- $(4+3i) \div (4+i) = \dfrac{(4+3i)(4-i)}{17} = \dfrac{19}{17} + \dfrac{8}{17} i.$

- $(4+3i) \div (4-3i) = \dfrac{(4+3i)^2}{25} = \dfrac{7}{25} + \dfrac{24}{25} i.$

- $\dfrac{i}{-i} = -1$