We can regard $\mathbb{Z}$ as a subset of $\mathbb{Q}$ by saying that $n = \frac{n}{1}$ for $n \in \mathbb{Z}$. Then we can regard the rational number $\frac{a}{b}$ as $a \div b$.

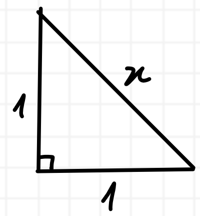
$\mathbb{Q}$ has an ordering $<$ which extends the ordering on $\mathbb{Z}$ and satisfies familiar rules. But there are important differences:

- There is no smallest positive rational number (Theorem 3.8). This means that we can't do proof by induction in $\mathbb{Q}$: what would the base case be?
- Given any rational numbers a and b with $a < b$, we can find a rational number between them (such as $\frac{a+b}{2}$). So there are no consecutive rational numbers.

8.2 Real numbers

In $\mathbb{Q}$ we can add, subtract, multiply and divide (except by 0). But there are equations we can't solve, such as $q^2 = 2$ (Theorem 3.7).

But there should be a number $\sqrt{2}$: consider the following triangle:



By Pythagoras's Theorem, the length of the hypotenuse satisfies $x^2 = 2$. So if we believe that the length of a line should be a number, then we need a number $\sqrt{2}$. So we need to extend our number system again.

There are various ways to define the real numbers. We'll use decimal expansions.

Let's think about what decimal expansions of rational numbers mean:

0.375 means $\frac{3}{10} + \frac{7}{100} + \frac{5}{1000} = \frac{3}{8}$.

Some rationals have an infinite decimal expansion:

0.11111.... means $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

we can evaluate this using the formula for the sum of a geometric series:

$$\frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{1}{9}.$$

$$0.18181818\ldots = \frac{18}{100} + \frac{18}{10000} + \frac{18}{1000000} + \cdots$$

$$= \frac{\frac{18}{100}}{1 - \frac{1}{100}} = \frac{18}{99} = \frac{2}{11}.$$

$$0.001001001001\ldots = \frac{1}{999}.$$

shortcut: $0.a_1a_2\ldots a_r\,a_1a_2\ldots a_r\,a_1a_2\ldots a_r\ldots$

$$= \frac{a_1a_2\ldots a_r}{99\ldots9}.$$

Some decimals don't repeat straight away:

$0.16666\ldots$

we can write this as

$$0.6666\ldots - 0.5$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

$0.5833333\ldots$

$$= 0.3333\ldots + 0.25$$

$$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

In fact, every rational number has a decimal expansion which is eventually periodic: it settles into a repeating pattern. Also, every eventually periodic decimal expansion is the decimal expansion of a rational number.

Def^n: A real number is an infinite decimal, i.e. an expression $n \cdot a_1 a_2 a_3 \ldots$ where $n \in \mathbb{Z}$ and $a_1, a_2, a_3, \ldots \in \{0, 1, \ldots, 9\}$.

We write $\mathbb{R}$ for the set of real numbers. Elements of $\mathbb{R} \setminus \mathbb{Q}$ are called irrational numbers.

Do different decimal expansions give different real numbers? Not quite: what is

$$0.99999\ldots \quad ?$$

$$0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

$$= \frac{9/10}{1-1/10} = 1.$$

In fact, any decimal ending in $9999\ldots$ equals a terminating decimal:

$$0.4379999\ldots = 0.438.$$

Apart from this, different decimals do give different numbers.

In $\mathbb{R}$ we can still add, subtract, multiply and divide (except by 0), and we have an order $<$, and these satisfy familiar rules.

But can we solve the equation $x^2 = 2$ in $\mathbb{R}$? We will construct $\sqrt{2}$ by working out its decimal expansion one digit at a time, using the fact that if $a, b > 0$, then $a < b \Leftrightarrow a^2 < b^2$.

What comes before the decimal point in $\sqrt{2}$?

$1^2 < 2 < 2^2$, so $1 < \sqrt{2} < 2$

so $\sqrt{2} = 1.???$

Next digit:

$$\underset{\substack{\| \\ 1.96}}{(1.4)^2} < 2 < \underset{\substack{\| \\ 2.25}}{(1.5)^2} \quad \text{so} \quad 1.4 < \sqrt{2} < 1.5$$

so $\sqrt{2} = 1.4???$

Next digit:

$(1.41)^2 < 2 < (1.42)^2$ so $1.41 < \sqrt{2} < 1.42$

so $\sqrt{2} = 1.41???$

We can continue like this an find an infinite decimal $\sqrt{2} = 1.41421356 2\ldots$ To show that this number satisfies $x^2 = 2$ requires some calculus.

So extending from $\mathbb{Q}$ to $\mathbb{R}$ allows us to solve $x^2 = 2$. We can solve other equations like $x^2 = 5$, $x^3 = 4$, $2^x = 6$, $x^2 = x + 1$. In fact, any equation where we can get very close to a solution in $\mathbb{Q}$ we can solve in $\mathbb{R}$.

8.3 Upper bounds

Now we introduce an important property that $\mathbb{R}$ has but $\mathbb{Q}$ doesn't.

Let's start with the idea of a maximum.

If X is a finite non-empty set of real numbers, then X has a maximum: an element of X which is bigger than all the other elements of X. We write this as $\max X$.

e.g. $\max\{1, 2, -\sqrt{2}, \pi, 10\} = 10$.

Some infinite sets also have a maximum.

e.g. $\max\{n \in \mathbb{Z} : n < 0\} = -1$.

$\max\{x \in \mathbb{R} : x^2 \leq 4\} = 2$

$\max[0,1] = 1.$

But many infinite sets don't have a maximum:

- $\mathbb{Z}$ has no maximum
- $[0,\infty)$ has no maximum
- $[0,1)$ has no maximum.

The last example is a bit different from the other two: $\mathbb{Z}$ and $[0,\infty)$ have no maximum because their elements get bigger and bigger. But $[0,1)$ has what looks like a maximum, namely 1, but $1 \notin [0,1)$. We'll introduce the idea of a supremum to deal with cases like this.

Defn: Suppose $X \subseteq \mathbb{R}$. An <u>upper bound</u> for X is a real number u such that $x \leq u$ for all $x \in X$. X is <u>bounded above</u> if it has an upper bound.

e.g.
- $\{1, 6, -3, -2\}$ is bounded above: 6 is an upper bound.
- More generally, any finite set is bounded above.
- $\emptyset$ is bounded above: 712 is an upper bound.
- $[0,1]$ is bounded above: 1 is an upper bound.
- More generally, any set that has a maximum is bounded above: $\max X$ is an upper bound.
- $[0,1)$ is bounded above: 1 is an upper bound.
- $\mathbb{Z}$ is not bounded above: there is no $u \in \mathbb{R}$ such that $x \leq u$ for every integer x.
- Similarly $[0,\infty)$ is not bounded above.

Any non-empty subset of $\mathbb{Z}$ which is bounded above has a maximum. But the same is not true in $\mathbb{Q}$ or in $\mathbb{R}$, because $[0,1)$ has no maximum.

Defⁿ: Suppose $X \subseteq \mathbb{R}$. A <u>supremum</u> for X is a real number s such that:

- s is an upper bound for X
- if t is any other upper bound for X, then $t \geq s$.

So a supremum is an upper bound for X which is smaller than any other upper bound for X. A supremum is also called a <u>least upper bound</u>.

e.g.

- $X = \{1, 2, 6\}$. Then 6 is a supremum for X: 6 is an upper bound because $1 \leq 6$, $2 \leq 6$, $6 \leq 6$. If t is any other upper bound, then in particular $6 \leq t$.
- More generally, if X has a maximum, then $\max X$ is a supremum.
- $X = [0,1)$. Then 1 is a supremum: obviously 1 is an upper bound. To show that 1 is a supremum, we need to show that any other upper bound is greater than 1; in other words, if $t < 1$ then t is not an upper bound for X. Given $t < 1$, choose x such that $t < x < 1$. Then $x \in X$, but $x > t$, so t is not an upper bound for X.
- $\mathbb{Z}$ has no supremum, because it has no upper bound.
- $\emptyset$ has no supremum, because <u>any</u> real number is an upper bound for $\emptyset$, so there is no least upper bound.

Lemma 8.1: Suppose $X \subseteq \mathbb{R}$, and X has a supremum.
Then X has a unique supremum.

Pf: Suppose s and t are both suprema for X. Then s and t are both upper bounds for X.

$s \leq t$ because s is a supremum and t is an upper bound
$t \leq s$ because t is a supremum and s is an upper bound.
So $s = t$. $\square$

In view of Lemma 8.1, we can talk about <u>the</u> supremum of a set. If X has a supremum, then we write $\sup X$ for the supremum of X.
We also write $\sup X = \infty$ if X is not bounded above.
We also write $\sup \emptyset = -\infty$.

Now we show why we need the real numbers.
Let $X = \{x \in \mathbb{Q} : x^2 \leq 2\}$.

Then X is bounded above, e.g. 2 is an upper bound.
X doesn't have a maximum: to show this, we have to show that given any $x \in X$, we can find $y \in X$ such that $y > x$.