Now we'll look at some properties of sequences of real numbers

Def^n: A sequence $(a_k)_{k\in\mathbb{N}}$ of real numbers is:

- <u>increasing</u> if $a_k < a_{k+1}$ for all $k \in \mathbb{N}$
- <u>decreasing</u> if $a_k > a_{k+1}$ for all $k \in \mathbb{N}$
- <u>weakly increasing</u> if $a_k \leq a_{k+1}$ for all $k \in \mathbb{N}$
- <u>weakly decreasing</u> if $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$
- <u>constant</u> if $a_k = a_{k+1}$ for all $k \in \mathbb{N}$.

"weakly increasing" is often called "non-decreasing".

Warning: Some books use "strictly increasing" and "increasing" in place of "increasing" and "weakly increasing".

We can add the word "eventually" before any property of sequences to mean that the sequence has that property after a certain point.

So $(a_k)_{k\in\mathbb{N}}$ is <u>eventually increasing</u> if there is $n\in\mathbb{N}$ such that $a_k < a_{k+1}$ for all $k > n$.

Examples:

- The sequence $\left(\frac{1}{k^2}\right)_{k\in\mathbb{N}}$:

  $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$

  This is a decreasing sequence.

- The Fibonacci sequence:

  $1, 1, 2, 3, 5, 8, 13, \dots$

  This sequence is weakly increasing and eventually increasing, but not increasing.

- The sequence $\left(\frac{(-1)^k}{k}\right)_{k\in\mathbb{N}}$

$$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \ldots$$
$$< \quad > \quad < \quad > \quad < \quad >$$

This is not weakly increasing or eventually increasing or weakly decreasing or eventually decreasing.

It does have a decreasing subsequence

$$\frac{1}{2} > \frac{1}{4} > \frac{1}{6} > \cdots$$

and an increasing subsequence

$$-1 < \frac{-1}{3} < \frac{-1}{5} < \cdots$$

- The sequence $((k-4)^2)_{k\in\mathbb{N}}$:

$$9, 4, 1, 0, 1, 4, 9, \ldots$$

This is not weakly increasing but is eventually increasing.

- The sequence $\left(\left\lfloor \frac{10}{k} \right\rfloor\right)_{k\in\mathbb{N}}$

remember: $\lfloor x \rfloor$ is the integer part of x, i.e. x rounded down to an integer.

$$10, 5, 3, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0, 0, \ldots$$

This sequence is weakly decreasing and eventually constant.

- The sequence of digits of π:

$$1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots$$

This sequence is not eventually increasing. It doesn't have an increasing subsequence: in an increasing sequence all the terms are different, but here there are at most 10 possible different terms.

The sequence must have a constant subsequence: there are only ten possible digits, so some digit must appear infinitely many times.

The sequence is not eventually constant, because π is irrational.

8. Rational and real numbers

8.1 Rational numbers

We extended $\mathbb{N}$ to $\mathbb{Z}$ to make subtraction possible. But we can't always divide in $\mathbb{Z}$: there is no integer n such that $3n = 2$, so we can't divide 2 by 3.

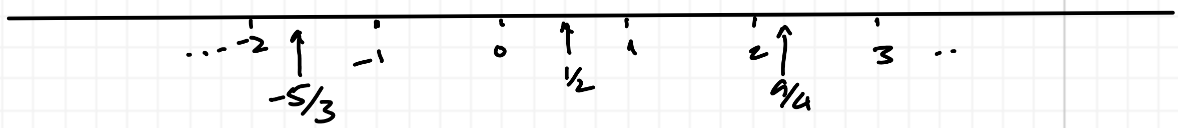
We introduce the rational numbers to allow division.

Defn: A <u>rational number</u> is an expression $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. We regard $\frac{a}{b}$ and $\frac{c}{d}$ as the same rational number whenever $ad = bc$.

For example $\frac{1}{2} = \frac{101}{202}$ because $1 \times 202 = 2 \times 101$.

We write $\mathbb{Q}$ for the set of rational numbers.

We can visualise the rational numbers on a number line:



Addition, subtraction and multiplication can be extended to $\mathbb{Q}$:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}, \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

These operations satisfy the same rules as before.

We can also divide:

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

<u>But</u> we can't divide by 0.

<u>Why not divide by 0?</u>

Why don't we introduce a new number ∞ satisfying $\frac{1}{0} = \infty$? Our rules for multiplication and division would break down.

Two rules that we would like to keep are:

- if $a = \frac{b}{c}$, then $ac = b$.
- $(a \times b) \times c = a \times (b \times c)$.

But if these rules apply in a number system with $\frac{1}{0} = \infty$, then

$\infty \times 0 = 1$. But then what is

$$\infty \times 0 \times 2 \; ?$$

On the one hand $(\infty \times 0) \times 2 = 1 \times 2 = 2$

on the other hand, $\infty \times (0 \times 2) = \infty \times 0 = 1$.

So we obtain $2 = 1$. $\lightning$.