Warning: Even though we write f^{-1}, we are not assuming f is invertible. We can always write f^{-1} if we apply it to subsets of B.

Examples

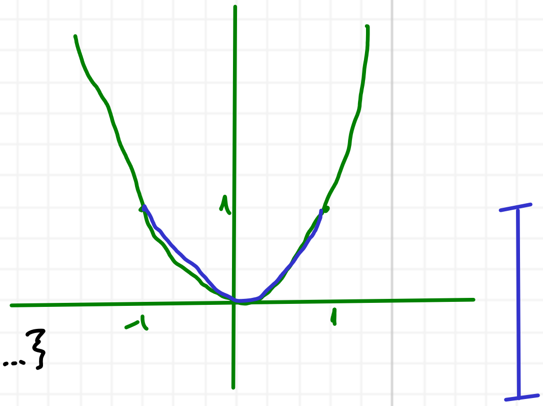
- $f : \{1,2,3\} \to \{\text{red}, \text{blue}, \text{green}\}$ given by

$$f(1) = \text{red}, \quad f(2) = \text{green}, \quad f(3) = \text{red}.$$

Then
$$f^{-1}(\{\text{green}\}) = \{2\}$$
$$f^{-1}(\{\text{red}\}) = \{1,3\}$$
$$f^{-1}(\{\text{red}, \text{blue}\}) = \{1,3\}$$
$$f^{-1}(\{\text{blue}\}) = \emptyset.$$

- $f : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$.

$$f^{-1}(\{1\}) = \{-1, 1\}.$$
$$f^{-1}(\{-1\}) = \emptyset$$
$$f^{-1}([-1,1]) = [-1,1]$$



$$f^{-1}(\mathbb{Z}) = \{0, 1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, \ldots\}$$

$$f^{-1}([9,\infty)) = (-\infty, -3] \cup [3, \infty)$$

- If f is any function, then

$$f^{-1}(B) = A, \qquad f^{-1}(\emptyset) = \emptyset.$$

Inverse images behave well with respect to set operations: If $C, D \subseteq B$, then

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$$

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$$

$$f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D).$$

(to see why the last one is true:

$$a \in f^{-1}(C \setminus D) \iff f(a) \in C \setminus D$$
$$\iff f(a) \in C \text{ and } f(a) \notin D$$
$$\iff a \in f^{-1}(C) \text{ and } a \notin f^{-1}(D)$$
$$\iff a \in f^{-1}(C) \setminus f^{-1}(D).$$
)

Another warning: images and inverse images of subsets are <u>not</u> inverses of each other: if $f: A \to B$ and $C \subseteq A$ and $D \subseteq B$, then $f^{-1}(f(C))$ is not necessarily equal to C, $f(f^{-1}(D))$ is not necessarily equal to D.

e.g. $f: \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$.

Let $C = [0,2]$.
Then $f(C) = [0,4]$.
So $f^{-1}(f(C)) = [-2,2] \neq C$.

Let $D = [-1,2]$
Then $f^{-1}(D) = [-\sqrt{2}, \sqrt{2}]$
So $f(f^{-1}(D)) = [0,2] \neq D$.

7. Some other mathematical objects

7.1 Relations

Def^n: Suppose X is a set. A <u>relation</u> on X is a property which may or may not hold for every ordered pair of elements of X.

We think of a relation as a symbol that we put between any two elements of X to give a statement. For example, $<$ is a relation on $\mathbb{Z}$: for any $a,b \in \mathbb{Z}$, we obtain a statement $a<b$ which is true or false.

More examples:

- If X is any set of numbers, then $\leq$ is a relation on X.
- If $X = \mathbb{N}$, then $\mid$ is a relation on X.
- If X is any set of sets, then $\subset$ is a relation on X.
- If X is <u>any</u> set, then $=$ is a relation on X.
- If X is any set, then $\neq$ is a relation on X.
  (In general, given any relation R on a set X, we get a new relation $\not R$, where $a \not R b$ means "not aRb".)
- If X is a set of sets, the relation R defined by

ARB if A and B are disjoint.

- The relation R on $\mathbb{Z}$ defined by aRb if $a-b$ is divisible by 3.

- If X is a set of people, then "loves" is a relation on X.

Def^n: Suppose R is a relation on a set X. R is:

- reflexive if aRa for all $a \in X$.
- symmetric if $aRb \Rightarrow bRa$, for all $a,b \in X$.
- anti-symmetric if there do not exist $a,b \in X$ such that $a \neq b$, aRb and bRa.
- transitive if $(aRb$ and $bRc) \Rightarrow aRc$, for all $a,b,c \in X$.

Examples

- The relation $>$ on $\mathbb{R}$. This is:
  - not reflexive: $1 \not> 1$.
  - not symmetric: $3 > 2$ but $2 \not> 3$
  - anti-symmetric: there do not exist $a,b \in \mathbb{R}$ such that $a > b > a$.
  - transitive: if $a > b$ and $b > c$, then $a > c$.

- The relation $\geqslant$ on $\mathbb{R}$. This is:
  - reflexive: $a \geqslant a$ for every $a \in \mathbb{R}$.
  - not symmetric (as above)
  - anti-symmetric: if $a \geqslant b$ and $b \geqslant a$, then $a = b$.
  - transitive (as above).

- The relation $\neq$ on $\mathbb{N}$. This is:
  - not reflexive: it is not true that $1 \neq 1$.
  - symmetric: if $a \neq b$ then $b \neq a$.
  - not anti-symmetric: $2 \neq 3$ and $3 \neq 2$
  - not transitive: $3 \neq 4$ and $4 \neq 3$, but it is not true that $3 \neq 3$.

- The relation $\mid$ on $\mathbb{N}$. This is:
  - reflexive: $a \mid a$ for every a.

not symmetric: $2 \mid 6$ but $6 \nmid 2$

antisymmetric: if $a \mid b$ and $b \mid a$, then in particular $a \le b \le a$, so $a = b$.

transitive (Lemma 4.1).

- The relation R on $\mathcal{P}(\mathbb{N})$ defined by ARB if $A \subseteq B$ or $B \subseteq A$. This is

  reflexive: $A \subseteq A$ for every A.

  symmetric: if $A \subseteq B$ or $B \subseteq A$, then $B \subseteq A$ or $A \subseteq B$.

  not antisymmetric: $\{1\} R \{1,2\}$ and $\{1,2\} R \{1\}$.

  not transitive: $\{1,2,3\} R \{1,2,3,4,5\}$ and $\{1,2,3,4,5\} R \{4,5\}$, but $\{1,2,3\} \not R \{4,5\}$.

- The relation R on $\mathbb{Z}$ defined by aRb if $3 \mid a-b$. R is:

  reflexive: $3 \mid a-a$ for every a.

  symmetric: if $3 \mid a-b$, then $3 \mid b-a$.

  not anti-symmetric: $3R6$ and $6R3$.

  transitive: if aRb and bRc, then $3 \mid a-b$ and $3 \mid b-c$, so $a-b = 3k$, $b-c = 3l$ where $k, l \in \mathbb{Z}$ so $a-c = 3(k+l)$, so $3 \mid a-c$, so aRc.

- The relation R on $\mathbb{Z}$ defined by aRb if $a=3$. R is:

  not reflexive: $2 \not R 2$.

  not symmetric: $3R5$ but $5 \not R 3$

  anti-symmetric: if aRb and bRa, then $a = 3 = b$.

  transitive: if aRb and bRc, then $a=3$, so aRc.

- Let X be the set of all people, and R the relation "is a brother of".

  not reflexive: no-one is their own brother

  not symmetric: King Charles R Princess Anne, but Princess Anne $\not R$ King Charles.

  not anti-symmetric: Charles R Edward and Edward R Charles.

  transitive: if a is b's brother and b is c's brother, then a is c's brother.

Some combinations of these properties are important: a relation which is reflexive, symmetric and transitive is called an <u>equivalence relation</u>. A relation which is reflexive, anti-symmetric and transitive is called a <u>partial order</u>.

7.2 Sequences

Defn: A <u>sequence</u> is an ordered list

$a_1, a_2, a_3, \dots$

of elements of some set X.

We will always take sequences to be infinite.

We sometimes write the sequence

$a_1, a_2, a_3, \dots$

as $(a_k)_{k \in \mathbb{N}}$ or $(a_k)_{k=1}^{\infty}$.

We can specify a sequence by giving enough terms to make the pattern obvious, or giving a formula for a_k in terms of k.

Examples:

- The sequence of real numbers

  $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$

  can be written as $(a_k)_{k \in \mathbb{N}}$ where $a_k = \frac{1}{k^2}$.

- The sequence

  $-1, 1, -1, 1, -1, 1, \dots$

  can be written as $(a_k)_{k \in \mathbb{N}}$, where $a_k = (-1)^k$.

- The sequence of sets

  $\{1,2\}, \{2,3,4\}, \{3,4,5,6\}, \{4,5,6,7,8\}, \dots$

  can be written as $(C_k)_{k \in \mathbb{N}}$, where $C_k = \{k, k+1, \dots, 2k\}$.

- The Fibonacci sequence
  $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$
  can be written as $(F_k)_{k \in \mathbb{N}}$, where
  $F_1 = F_2 = 1, \quad F_k = F_{k-1} + F_{k-2}$ if $k \geq 3$.

- The sequence
  $1, 4, 1, 5, 9, 2, 3, 5, 9, \ldots$
  is the sequence of digits after the decimal point in π.

Defn. Suppose S is a sequence. A <u>subsequence</u> of S is a sequence obtained by deleting some of the terms of S, keeping the others in order.

(Remember that sequences are infinite, so when we delete terms we must leave infinitely many terms.)

Examples:

- The sequence $(2^k)_{k \in \mathbb{N}}$:

  $2, 4, 8, 16, 32, 64, 128, \ldots$

  Deleting the first two terms gives the subsequence

  $8, 16, 32, 64, 128, \ldots$.

  This is the sequence $(b_k)_{k \in \mathbb{N}}$, where $b_k = 2^{k+2}$.

  If instead we delete the odd-numbered terms, we get

  $4, 16, 64, 256, \ldots$.

  This is the sequence $(4^k)_{k \in \mathbb{N}}$.

- The sequence $((-1)^k)_{k \in \mathbb{N}}$:

  $-1, 1, -1, 1, \ldots$

  if we delete the first two terms, we still have the same sequence.

  if we delete the odd-numbered terms, we get

  $1, 1, 1, 1, \ldots$

  i.e. the constant sequence with value 1.

- The Fibonacci sequence

  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

  If we delete the first three terms, we get the subsequence

  3, 5, 8, 13, 21, 34, ...

  This sequence can be defined by

  $G_1 = 3, \quad G_2 = 5, \quad G_k = G_{k-1} + G_{k-2} \quad \text{for } k \geq 3.$

  Now take the subsequence of the Fibonacci sequence consisting of the even terms:

  2, 8, 34, 144, 610, ...

  This sequence can be defined by

  $G_1 = 2, \quad G_2 = 8, \quad G_k = 4G_{k-1} + G_{k-2}.$
