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The primes are

2, 3, 5, 7, 11, 13, 17, 19, 23, ...

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Fundamental Theorem of Arithmetic: The prime factorisation is unique up to re-ordering.

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Greatest common divisors

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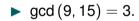
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Examples

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- gcd(a, a) = a for any a.
- ▶ If $b \mid a$, then gcd (a, b) = b.

Finding gcd(a, b)

Write down all the divisors of *a* and *b*, and find the highest number in both lists.

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1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

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This method is very slow.

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So

$$gcd(100, 120) = 2 \times 2 \times 5 = 20.$$

Finding gcd(a, b)

Finding gcd(a, b) – fast method

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Proposition 4.6: If $a, b \in \mathbb{N}$ and $q, r \in \mathbb{Z}$ with 0 < r < b and a = qb + r, then gcd(a, b) = gcd(b, r).